MATHEMATICS

ACCOUNTING PROFESSION OPTION

For Rwandan Schools

Senior **5** Student Book

EXPERIMENTAL VERSION

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FOREWORD

Dear Student,

Rwanda Basic Education Board (REB) is honoured to present Senior 5 Mathematics book for the students of Accounting Profession Option which serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the Mathematics. The Rwandan educational philosophy is to ensure that you achieve full potential at every level of education which will prepare you to be well integrated in society and exploit employment opportunities.

The government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate your learning process. Many factors influence what you learn, how well you learn and the competences you acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials available. In this book, we paid special attention to the activities that facilitate the learning process in which you can develop your ideas and make new discoveries during concrete activities carried out individually or in groups.

In competence-based curriculum, learning is considered as a process of active building and developing knowledge and meanings by the learner where concepts are mainly introduced by an activity, situation or scenario that helps the learner to construct knowledge, develop skills and acquire positive attitudes and values.

For efficiency use of this textbook, your role is to:

- Work on given activities which lead to the development of skills;
- Share relevant information with other learners through presentations, discussions, group work and other active learning techniques such as role play, case studies, investigation and research in the library, on internet or outside;
- Participate and take responsibility for your own learning;
- Draw conclusions based on the findings from the learning activities.

To facilitate you in doing activities, the content of this book is self-explanatory so that you can easily use it yourself, acquire and assess your competences. The book is made of units as presented in the syllabus. Each unit has the following structure: the unit title and key unit competence are given and they are followed by the introductory activity before the development of mathematical concepts that are connected to real world problems more especially to production, finance and economics.

The development of each concept has the following points:

- Learning activity which is a well set and simple activity to be done by students in order to generate the concept to be learnt;
- Main elements of the content to be emphasized;
- Worked examples; and
- Application activities to be done by the user to consolidate competences or to assess the achievement of objectives.

Even though the book has some worked examples, you will succeed on the application activities depending on your ways of reading, questioning, thinking and handling calculations problems not by searching for similar-looking worked out examples.

Furthermore, to succeed in Mathematics, you are asked to keep trying; sometimes you will find concepts that need to be worked at before you completely understand. The only way to really grasp such a concept is to think about it and workrelated problems found in other reference books.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, development partners, Universities Lecturers and secondary school teachers for their technical support. A word of gratitude goes to Secondary Schools Head Teachers, Administration of different Universities (Public and Private Universities) and development partners who availed their staff for various activities.

Any comment or contribution for the improvement of this textbook for the next edition is welcome.

Dr. MBARUSHIMANA Nelson

Director General, REB.

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I wish to express my appreciation to the people who played a major role in the development of this Mathematics book for Senior Five students in Accounting Profession Option. It would not have been successful without active participation of different education stakeholders.

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I wish to extend my sincere gratitude to Universities Lecturers, Secondary school teachers and staff from different education parterres whose efforts during writing exercise of this book were very much valuable.

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MURUNGI Joan

Head of CTLR Department

TABLE OF CONTENT

Unit 1 Matrices and determinants

Key Unit competence: Use matrices and determinants notations and properties to solve simple production, financial, economical, and mathematical related problems.

Introductory activity

The table below shows the revenue and expenses (in Rwandan francs) of a family over three consecutive months:

	October	November	December
Revenue	450,000	460,000	700,000
Expenses	440,000	295,000	890,000

a) What was the family's revenue in October?

b) By how much money did the family's revenue increase from October to November?

1.1 Generalities on matrices

1.1.1. Definitions and notations

Learning Activity 1.1.1

A shop selling shirts records the number of each type of shirts it sells over a period of two weeks. In the first week, it sells 12 small size shirts, 8 medium shirts and 5 large shirts.

In the second week, it sells only 9 small size shirts and 3 medium size shirts.

a) What are the two criteria the shopkeeper will use to record these data?

b) Record this information in a rectangular array consisting of double entries.

c) Such a table is called a "matrix". Describe the components of a matrix.

CONTENT SUMMARY

– A matrix is a rectangular arrangement of numbers, in rows and columns, with$_i$n brackets $(\ )$ or $[\ ]$. A matrix is denoted by a capital letter: A, B, C, ...

Rows are counted from the top of the matrix to the bottom of the matrix; columns are counted from the leftmost side of the matrix to the rightmost side of the matrix.

– The numbers in the matrix **are called entries** or **elements.**

The position of an entry in the matrix is shown by lower subscripts, such as a_j :the entry on the ith row and jth column.

– If matrix A has n rows and p columns, then we say that the matrix A is of **order** $n \times p$**,** read n by p, where the product $n \times p$ is the number of entries in the matrix.

Note: In finding the order of a matrix, we do not perform the multiplication $n \times p$**,** we just write $n \times p$**,** but for finding the number of entries of a matrix given by its order $n \times p$,we calculate the product $n \times p$**.**

If A is a matrix of order $n \times p$, then A can generally be written as $A = (a_{ij})$,where i and j are positive integers, and; $1 \le i \le n ; 1 \le j \le p$.

– A matrix with only one row is said to be a **row matrix;** that is a matrix of order $1 \times p$.

Thus, $\begin{pmatrix} 2 & 4 & 7 \end{pmatrix}$ is a row matrix.

A matrix with only one column is said to be a **column matrix;** that is a matrix of order $n \times 1$.

Thus, $\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$ is a column matrix.

– A **square matrix** is a matrix in which the number of rows is equal to the number of columns; that is, matrix A of order $n \times p$ is a square matrix if and only if $n = p$;

In this case, instead of saying a matrix of order $n \times n$, we, sometimes, simply say a matrix of order n.

Thus, $A=\begin{pmatrix}3 & -2\\ 1 & 0\end{pmatrix}$; $B=\begin{pmatrix}4 & 5 & 2\\ 2 & 0 & 3\\ 1 & 7 & 6\end{pmatrix}$ are square matrices of orders 2 and 3, respectively.

If $A=(a_{ij})$ is a square matrix of order 2, then i and j assume values in the set $\{1,2\}$. Therefore, $(a_{ij})=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$.

In the same way, if $A=(a_{ij})$ is a square matrix of order 3, then i and j assume values in the set $\{1,2,3\}$. Therefore, $(a_{ij})=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$.

Example 1.1.1.

In the matrix $M=(a_{ij})=\begin{pmatrix}1 & 3 & 4\\ 5 & 12 & 13\\ 7 & 6 & 0\end{pmatrix}$,

a) Write down the value of a_{23} and the value of a_{31}

b) Explain why the matrix is a square matrix

Solution:

a) The entry on the second row and third column is $a_{23}=13$;

The entry on the third row and first column is $a_{31}=7$.

b) M is a square matrix because it has the same number of rows and columns

Application activity 1.1.1

1. Write down the order of each of the following matrices:

   a) $A=\begin{pmatrix}8 & 6 & 2\\ -1 & 1 & 0\end{pmatrix}$ b) $B=\begin{pmatrix}-2 & 1 & 3\end{pmatrix}$ c) $C=\begin{pmatrix}7\\ -5\end{pmatrix}$

2. A shoe shop sells shoes for men and ladies. The first week, it sold 7 pairs of men's shoes and 15 pairs of ladies' shoes. The second week, it sold 9 pairs of ladies 'shoes and 4 pairs of men's shoes. Record this information as a 2×2 matrix, stating what the rows stand for, and what the columns stand for.

1.1.2. Equality of matrices

Learning Activity 1.1.2

Consider the following situations:

Situation1:

A class consists of boys and girls who are boarders or day scholars. The class teacher records the data by the matrix

$A=\begin{pmatrix}8 & 9\\ 17 & 6\end{pmatrix}$, where the columns represent the numbers of boys and girls, and the rows represent the numbers of boarders and day scholars.

Situation2:

Two brothers sell shirts and shoes, in two different shop I and II, for two consecutive weeks. The Elder brother records his data by the matrix

$B=\begin{pmatrix}8 & 9\\ 17 & 6\end{pmatrix}$, where the columns represent the numbers of shirts and shoes, and the rows represent the numbers of items sold in week1, and in week2.

The younger brother, also, records his data by the matrix $C=\begin{pmatrix}8 & 9\\ 17 & 6\end{pmatrix}$, where the columns represent the numbers of shirts and shoes, and the rows represent the numbers of items sold in week1, and in week2. Comment on the following, for matrices A, B and C:

a) Number of rows and columns

b) Corresponding entries (that is entries occupying the same positions)

c) Nature of the elements.

d) Predict which two of the matrices above (A, B and C) are equal.

e) What are the conditions for two matrices to be equal?

CONTENT SUMMARY

Two matrices $A=(a_{ij})$ and $B=(b_{ij})$ are **equal** if and only if:

i) they have the same order;

ii) the corresponding entries (that is the entries occupying the same position, in terms of rows and columns) are equal.

iii) The nature of the entries in the two matrices is the same.

Note: When discussing the equality of matrices, we assume the nature of the entries in the two matrices to be the same.

Thus, matrices $A=\begin{pmatrix} 3 & 0 \\ -1 & \frac{-18}{-9} \\ \sqrt{16} & 2^3 \end{pmatrix}$ and $B=\begin{pmatrix} 2+1 & 7-7 \\ \frac{13}{-13} & 2 \\ 4 & 8 \end{pmatrix}$ are equal.

Example1.1.2.

Given that matrices $A=\begin{pmatrix} 2x-1 & -3 \\ 2 & 2x+y \end{pmatrix}$ and $B=\begin{pmatrix} 9 & -3 \\ 2 & -11 \end{pmatrix}$ are equal, find the values of x and y.

Solution:

Matrices A and B are both of order 2×2 and for the corresponding entries to be equal, we have:

$\begin{cases} 2x-1=9 \\ 2x+y=-11 \end{cases}$. Solving simultaneously, we get $\begin{cases} x=5 \\ y=-21 \end{cases}$

Application activity 1.1.2

1. 1. Determine whether the following matrices A and B are equal or not:

a) $A=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ and $B=\begin{pmatrix}1 & 1\\ 1 & 1\\ 1 & 1\end{pmatrix}$ b) $A=\begin{pmatrix}5-4 & 3\\ 2 & \frac{12}{3}\end{pmatrix}$ and $B=\begin{pmatrix}1 & 2^3-5\\ \sqrt{9}-1 & 4\end{pmatrix}$

2. Given that matrices $X=\begin{pmatrix}2x^2-x & \frac{1}{3}\\ 0 & z\end{pmatrix}$ and $Y=\begin{pmatrix}1 & y\\ t^2+t & -5\end{pmatrix}$ are equal,

find the values of x, y, z and t.

1.2. Operations on matrices

1.2.1. Addition and subtraction of matrices

Learning Activity 1.2.1

A retailer sells two products, P and Q, in two shops, S and T.

She recorded the numbers of items sold for the last three weeks in each shop by the following matrices:

$$S=\begin{pmatrix}6 & 5 & 13\\ 11 & 13 & 10\end{pmatrix} \text{ and } T=\begin{pmatrix}7 & 8 & 2\\ 7 & 17 & 20\end{pmatrix}.$$

a) Write down the order of each of the two matrices S and T. How are these two orders?

b) Determine a single matrix for the total sales for this retailer for the last three weeks in the two shops.

c) Predict the conditions for two matrices to be added and how to obtain the sum of two matrices.

CONTENT SUMMARY

Matrices that have the same order can be added together, or subtracted. The addition, or subtraction, is performed on each of the corresponding elements.

Thus, if $A=(a_{ij})$ and $B=(b_{ij})$ are two matrices of the same order $n\times p$, then the sum of these matrices is the matrix $C=A+B$, of order $n\times p$,defined by $A+B=(a_{ij})+(b_{ij})=(a_{ij}+b_{ij})$, where i and j are positive integers, and $1\le i\le n; 1\le j\le p$

In the same way, subtracting matrix B from matrix A yields in matrix $D=A-B$, of order $n\times p$,defined by $A-B=(a_{ij})-(b_{ij})=(a_{ij}-b_{ij})$, where i and j are positive integers, and $1\le i\le n; 1\le j\le p$.

In particular, $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$, and

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{pmatrix}$$

Example 1.2.1.

Given matrices $P=\begin{pmatrix} 11 & 29 \\ 7 & 14 \end{pmatrix}$ and $Q=\begin{pmatrix} 6 & 34 \\ 3 & 7 \end{pmatrix}$, find the matrices:

a) $Q-P$

b) $P+Q$

Solution:

a) $Q-P=\begin{pmatrix} 6 & 34 \\ 3 & 7 \end{pmatrix}-\begin{pmatrix} 11 & 29 \\ 7 & 14 \end{pmatrix}=\begin{pmatrix} 6-11 & 34-29 \\ 3-7 & 7-14 \end{pmatrix}=\begin{pmatrix} -5 & 5 \\ -4 & -7 \end{pmatrix}$

b) $P+Q=\begin{pmatrix} 11 & 29 \\ 7 & 14 \end{pmatrix}+\begin{pmatrix} 6 & 34 \\ 3 & 7 \end{pmatrix}=\begin{pmatrix} 11+6 & 29+34 \\ 7+3 & 14+7 \end{pmatrix}=\begin{pmatrix} 17 & 63 \\ 10 & 21 \end{pmatrix}$

Application activity 1.2.1

1. Say, with reason, whether matrices A and B can be added or not. In case they can be added, find their sum and the difference $A-B$

a) $A=\begin{pmatrix}3 & 5\\ 8 & 1\end{pmatrix}$; $B=\begin{pmatrix}10 & 3 & 5\\ 11 & 11 & 4\end{pmatrix}$

b) $A=\begin{pmatrix}1 & 17\\ 14 & 3\end{pmatrix}$; $B=\begin{pmatrix}11 & 29\\ 8 & 6\end{pmatrix}$

2. In a sector of a district, there are three secondary schools, A, B and C having both boarding and day sections for both boys and girls. The distribution of the students in the three schools are given, respectively

by the matrices $A=\begin{pmatrix}350 & 500\\ 135 & 248\end{pmatrix}$; $B=\begin{pmatrix}420 & 312\\ 795 & 287\end{pmatrix}$; $C=\begin{pmatrix}124 & 0\\ 41 & 264\end{pmatrix}$,

where the first rows indicate the number of girls, the second rows the number of boys, the first columns the number of boarders and the second columns the number of day scholars in the three schools.

The Sector Education Officer (S.E.O) would like to record these data as a single matrix S.

a) Which operation should he/she perform on the three matrices to obtain matrix S?

b) Write down matrix S.

c) Use matrix S to answer the following questions:

i) How many day scholars are there from these three schools?

ii) How many girls are boarders from these three schools?

1.2.2. Scalar multiplication

Learning Activity 1.2.2

The monthly rental prices (in thousand Rwandan Francs) of three apartments without VAT (Value Added Tax) are recorded by the matrix below:

$M = \begin{pmatrix} 150 & 120 & 300 \end{pmatrix}$.

a) How do you calculate the VAT on an item?

b) What is the single operation to use in order to obtain the matrix M' representing the monthly rental prices of the three apartments, including 18% of VAT?

c) How do you obtain matrix M'?

d) Write down matrix M'

CONTENT SUMMARY

A matrix can be multiplied by a specific number; in this case, each entry of the matrix is multiplied by the given number. This type of multiplication is called scalar multiplication, since the matrix is multiplied by a single real number, and real numbers are also called scalars.

Thus, if $A = (a_{ij})$ of any order $n \times p$ and α a scalar, then $\alpha A = \alpha (a_{ij}) = (\alpha a_{ij})$, where i and j are positive integers, and $1 \le i \le n; 1 \le j \le p$

In particular, $\alpha \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{pmatrix}$, and

$$\alpha \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \\ \alpha a_{31} & \alpha a_{32} & \alpha a_{33} \end{pmatrix}$$

Example 1.2.2.

Given matrices $A = \begin{pmatrix} 1 & -2 \\ 7 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & 3 \\ 2 & 7 \end{pmatrix}$, find the matrix $2A + 3B$

Solution:

$$2A+3B=2\begin{pmatrix}1&-2\\7&4\end{pmatrix}+3\begin{pmatrix}6&3\\2&7\end{pmatrix}=\begin{pmatrix}2&-4\\14&8\end{pmatrix}+\begin{pmatrix}18&9\\6&21\end{pmatrix}=\begin{pmatrix}20&5\\20&29\end{pmatrix}$$

Application activity 1.2.2

1. Given matrices $A=\begin{pmatrix}3&4\\5&0\end{pmatrix}$ and $B=\begin{pmatrix}-2&1\\5&3\end{pmatrix}$,find:

   a) $4A-5B$

   b) $2(A+B)$

3) **2.** In November a shop sells clothes and shoes for both boys and girls.

The number of clothes and shoes sold for boys and girls are

recorded in the matrix $N=\begin{pmatrix}15&21\\19&28\end{pmatrix}$, where the number of boys and girls are in

columns, and the numbers of shoes and clothes are in rows.

Since the festive period of Christmas is approaching, the shop expects to double the number of each item to sell. Express the resulting matrix D.

1.2.3. Multiplication of matrices

Learning Activity 1.2.3

Two friends Agnes(A) and Betty(B) can buy sugar, rice and beans at one

or two supermarkets S_1 and S_2. Matrix $M=\begin{pmatrix}1&3&2\\2&4&3\end{pmatrix}$ shows the number

of kilograms of each of the three items(sugar, rice and beans)bought by

each of the two friends .Matrix $P=\begin{pmatrix}1800&1600\\2500&2000\\1500&1200\end{pmatrix}$ shows the price per

kilogram, in Rwandan francs, of sugar, rice and beans in supermarkets S_1

and S_2, respectively.

a) Compare the number of columns of M to the number of rows of P.

b) Calculate the shopping bill of each of the two friends at each of the two supermarkets. Express the answer as matrix C.

c) How many rows and how many columns does C have?

d) Use matrix M and P to explain how each entry of C is obtained.

CONTENT SUMMARY

Let A and B be matrices of order $n \times p$, and $m \times r$, respectively. Matrices A and B can be multiplied, in this order, if and only if $p = m$, that is, the number of columns of the first matrix is equal to the number of rows of the second matrix.

In this case, we say that matrices A and B, in this order, are **conformable for multiplication.** The product $A \times B$ is of order $n \times r$, that is the product $A \times B$ has the same number of rows as matrix A, and the same number of columns as matrix B.

Practically, we proceed as follows:

1. Determine the order of the product:

$A : n \times p$

$B : \quad p \times r$

$A.B : n \quad \times \quad r$

2. Calculate the entries of the product as follow: Let $A.B = (c_{ij})$. To determine the entry c_{ij}, multiply the entries along the ith row of the first matrix by the corresponding entries down the jth column of the second matrix and add the products.

In particular, if $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, then

$$A.B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} . \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

Example 1.2.3.

1. Consider matrices $A=\begin{pmatrix} 2 & 3 & 0 \\ -1 & 4 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} 3 & 1 \\ -2 & 4 \\ 2 & -1 \end{pmatrix}$

   Determine whether A and B, in any order, are conformable for multiplication or not.

   b) In case, they are conformable for multiplication, find the order of the products $A.B$ and $B.A$.What do you conclude about the multiplication of matrices?

   c) Find the matrix $A.B$

Solution:

a)

$A: 2\times 3$

$B: \quad 3\times 2$

$A.B: 2 \quad \times \quad 2$

A and B are conformable for multiplication, since the number of columns of A is equal to the number of rows of B

In the same way,

$\text{B}: 3\times 2$

$\text{A}: \quad 2\times 3$

$\text{B.A}: 3 \quad \times \quad 3$

B and A are conformable for multiplication, since the number of columns of B is equal to the number of rows of A.

b) The order of the product $A.B$ is 2×2, and the order of the product $B.A$ is 3×3.

Multiplication of matrices is **not commutative.** In general, for matrices A and B, $A.B \neq B.A$

c) $A.B=\begin{pmatrix}2&3&0\\-1&4&1\end{pmatrix}.\begin{pmatrix}3&1\\-2&4\\2&-1\end{pmatrix}=\begin{pmatrix}2(3)+3(-2)+0(2)&2(1)+3(4)+0(-1)\\-1(3)+4(-2)+1(2)&-1(1)+4(4)+1(-1)\end{pmatrix}=\begin{pmatrix}0&14\\-9&14\end{pmatrix}$

2. Consider matrices $A=\begin{pmatrix}1&3&-1\\-2&5&3\end{pmatrix}$, $B=\begin{pmatrix}5\\2\\4\end{pmatrix}$ and $C=\begin{pmatrix}2&-3\end{pmatrix}$.

a) Obtain the products $(A.B).C$ and $A.(B.C)$

b) Which property do you predict about multiplication of matrices?

Solution:

a) $A.B=\begin{pmatrix}1&3&-1\\-2&5&3\end{pmatrix}.\begin{pmatrix}5\\2\\4\end{pmatrix}=\begin{pmatrix}7\\12\end{pmatrix}$

$(A.B).C=\begin{pmatrix}7\\12\end{pmatrix}.\begin{pmatrix}2&-3\end{pmatrix}=\begin{pmatrix}14&-21\\24&-36\end{pmatrix}$

$B.C=\begin{pmatrix}5\\2\\4\end{pmatrix}.\begin{pmatrix}2&-3\end{pmatrix}=\begin{pmatrix}10&-15\\4&-6\\8&-12\end{pmatrix}$

$A.(B.C)=\begin{pmatrix}1&3&-1\\-2&5&3\end{pmatrix}.\begin{pmatrix}10&-15\\4&-6\\8&-12\end{pmatrix}=\begin{pmatrix}14&-21\\24&-36\end{pmatrix}$

We can predict that multiplication of matrices is **associative**, that is, for all matrices A, B and C, conformable for multiplication, $(A.B).C=A.(B.C)$

3. Let $A=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$,and $B=\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}$.

a) Find the product $A.B$

b) What do you conclude?

Solution:

a) $A.B=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}.\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$

A matrix, in which all the entries are zeros is said to be the **null matrix** or the **zero matrix.**

For matrices, the equality $A.B=0$ does not imply $A=0$ or $B=0$, that is , the product of two matrices can be the null matrix, yet none of the factors is a null matrix.

Application activity 1.2.3

1. Determine whether matrices A and B, in this order, are conformable for multiplication or not. In case, they are conformable, find the product:

   a) $A=\begin{pmatrix}3 & 1 & 12\end{pmatrix}$ and $B=\begin{pmatrix}4 & 2 & 1\end{pmatrix}$

   b) $A=\begin{pmatrix}5 & 2\end{pmatrix}$ and $B=\begin{pmatrix}3 & 2\\ 7 & 6\end{pmatrix}$

2. A company's input requirement over the next two months for two inputs X and Y are given (in numbers of units of each input) by the matrix $M=\begin{pmatrix}2 & 1\\ 6 & 8\end{pmatrix}$.

The company can buy these inputs from two suppliers, whose prices for the two inputs are given by the matrix $N=\begin{pmatrix}4 & 2\\ 5 & 1\end{pmatrix}$, where the two rows represent the suppliers and the columns represent the prices.

Obtain the matrix for the total input bill for the next two months for both suppliers.

1.2.4. Inversion of matrices

Learning Activity 1.2.4

Consider matrices $A=\begin{pmatrix}1 & 0 & 2\\ 0 & 1 & -3\\ 1 & -1 & 0\end{pmatrix}$ and $B=\frac{1}{5}\begin{pmatrix}3 & 2 & 2\\ 3 & 2 & -3\\ 1 & -1 & -1\end{pmatrix}$

a) Find the product $A.B$, and write down its order.

b) Describe the entries of the product.

c) If such a matrix is called identity matrix, which of the following are identity matrices: i) $\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ ii) (1) iii) $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ iv) $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

v) $\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 0\end{pmatrix}$

d) Matrix B is said to be the inverse of matrix A. When do we say that matrix X is inverse of matrix M?

CONTENT SUMMARY

– A square matrix with each element along **the main diagonal** (from the top left to the bottom right) being equal to 1 and with all other elements being 0 is said to be the identity matrix, it is denoted by **I;**

For any square matrix A of order $n\times n$, and the identity matrix I of order $n\times n$, we have:

$A.I=A$ and $I.A=A$, that is, I is the identity element for multiplication of matrices.

In particular,

Order	Matrix A	Identity I	Product $A.I = I.A = A$
1×1	(a)	(1)	$(a).(1)=(1).(a)=(a)$
2×2	$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a & c \\ b & d \end{pmatrix}.\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.\begin{pmatrix} a & c \\ b & d \end{pmatrix}=\begin{pmatrix} a & c \\ b & d \end{pmatrix}$
3×3	$\begin{pmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{pmatrix}.\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.\begin{pmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{pmatrix}=\begin{pmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{pmatrix}$

- If for a square matrix A of order $n\times n$, there exists a square matrix B of order $n\times n$, such that $A.B = I$ and $B.A = I$, where I is the identity matrix of order $n\times n$, then B is said to be the **inverse** of matrix A, and written $B = A^{-1}$
- To find the inverse of a square matrix A, of order $n\times n$, by **Gaussian method,** we, practically, proceed as follows:

Write (A/I), a matrix of order $n\times(2n)$, since the number of columns doubled, but the number of rows is unchanged. Matrix (A/I) is an **augmented matrix;**

Transform the matrix (A/I), using **elementary row operations,** to (I/B).

Then $B = A^{-1}$.

- The following are the elementary row operations:
  1. Interchanging two rows. For example, if row 1 and row 2 are interchanged, then the entries of row 1 become the respective entries of row2, and vice versa; we write $R_1 \leftrightarrow R_2$
  2. Multiplying each entry of a non-zero real number k. For example, if the entries of row3 are multiplied by, say 2, we write $R_3 \to 2R_3$
  3. Adding to each entry of a row any multiple from any **other** row, for example, $R_1 \to R_1 + kR_2$

If matrices B exists, then we say that A is **invertible** or **regular;**

If B does not exist, then we say that A is a **singular** matrix.

Example 1.2.4.

Use elementary row operations to find the inverse of matrix $A=\begin{pmatrix}1 & 2\\ 1 & -3\end{pmatrix}$, and verify that the product of the matrix A and its inverse is the identity matrix.

Solution:

Consider the augmented matrix $\begin{pmatrix}1 & 2 & 1 & 0\\ 1 & -3 & 0 & 1\end{pmatrix}$

$$\begin{pmatrix}1 & 2 & 1 & 0\\ 1 & -3 & 0 & 1\end{pmatrix} R_2 \to R_2 - R_1 \begin{pmatrix}1 & 2 & 1 & 0\\ 0 & -5 & -1 & 1\end{pmatrix};$$

$$\begin{pmatrix}1 & 2 & 1 & 0\\ 0 & -5 & -1 & 1\end{pmatrix} R_1 \to R_1 + \frac{2}{3}R_2 \begin{pmatrix}1 & 0 & \frac{3}{5} & \frac{2}{5}\\ 0 & -5 & -1 & 1\end{pmatrix};$$

$$\begin{pmatrix}1 & 0 & \frac{3}{5} & \frac{2}{5}\\ 0 & -5 & -1 & 1\end{pmatrix} R_2 \to -\frac{1}{5}R_2 \begin{pmatrix}1 & 0 & \frac{3}{5} & \frac{2}{5}\\ 0 & 1 & \frac{1}{5} & -\frac{1}{5}\end{pmatrix};$$

Therefore, the inverse matrix is $A^{-1}=\begin{pmatrix}\frac{3}{5} & \frac{2}{5}\\ \frac{1}{5} & -\frac{1}{5}\end{pmatrix}=\frac{1}{5}\begin{pmatrix}3 & 2\\ 1 & -1\end{pmatrix}$

The product is $A.A^{-1}=\begin{pmatrix}1 & 2\\ 1 & -3\end{pmatrix}.\frac{1}{5}\begin{pmatrix}3 & 2\\ 1 & -1\end{pmatrix}=\frac{1}{5}\begin{pmatrix}1 & 2\\ 1 & -3\end{pmatrix}.\begin{pmatrix}3 & 2\\ 1 & -3\end{pmatrix}$

$$=\frac{1}{5}\begin{pmatrix}5 & 0\\ 0 & 5\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Application activity 1.2.4

Given the matrix $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$,

a) Write down the identity matrix I of order 3×3 and determine the augmented matrix (M/I)

b) Use elementary row operations to transform (M/I) to (I/N)

c) Write down matrix M^{-1}

d) Verify that $M.M^{-1} = I$ and $M^{-1}.M = I$

1.3. Determinants of square matrices

1.3.1.Definition and calculation of determinants of matrices of orders 2×2 and 3×3

Learning Activity 1.3.1

1. Analyze the nature of entries in the rows (or columns) of the following square matrices $A = \begin{pmatrix} 4 & 1 \\ 5 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ 12 & 8 \end{pmatrix}$:

   a) In matrix B, how are the entries in row 2 obtained from the corresponding entries in row 1?

   b) Can you suggest a means for characterizing a square matrix as singular or not?

2. Determine whether matrix $A = \begin{pmatrix} 4 & 3 & 7 \\ 1 & 6 & 7 \\ 3 & 1 & 4 \end{pmatrix}$ is singular or not.

CONTENT SUMMARY

Let $A=\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ be a square matrix of order 2×2.Then the **determinant of** A is the unique real number denoted and defined by $\det A=\begin{vmatrix} a & c \\ b & d \end{vmatrix}=ad-bc$.

If $A=\begin{pmatrix} a & a' & a" \\ b & b' & b" \\ c & c' & c" \end{pmatrix}$ is a square matrix of order 3×3, then the determinant of A is the unique real number denoted and defined by $\det A=\begin{vmatrix} a & a' & a" \\ b & b' & b" \\ c & c' & c" \end{vmatrix}$

$=(ab'c"+a'b"c+a"bc')-(cb'a"+c'b"a+c"ba')$: a sum of six terms, each term having three factors

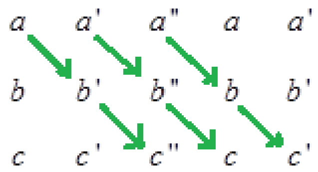
The following techniques can be used in evaluating the determinant of a square matrix $A=\begin{pmatrix} a & a' & a" \\ b & b' & b" \\ c & c' & c" \end{pmatrix}$ of order 3×3:

Sarrus' method

1. Copy the first two columns as the fourth and fifth columns:

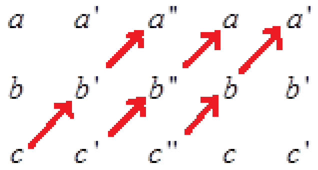
$$\begin{matrix} a & a' & a" & a & a' \\ b & b' & b" & b & b' \\ c & c' & c" & c & c' \end{matrix}$$

2. Multiply the entries along the descending arrows and add the products to obtain P_1



$P_1 = ab'c'' + a'b''c' + a''bc'$

Multiply the entries along the ascending arrows and add the products to obtain P_2



$P_2 = cb'a'' + c'b''a + c''ba'$

The value of the determinant is $P_1 - P_2$, that is,

$$\det A = \begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{vmatrix} = P_1 - P_2 = (ab'c'' + a'b''c + a''bc') - (cb'a'' + c'b''a + c''ba')$$

Expansion by cofactors

We have:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12})$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The determinant of the matrix remaining after deleting the row and the column of an entry is called the **minor** of that element. Thus, if M_{ij} is the matrix remaining after deleting the i^{th} row and the j^{th} column of a square matrix

$A = (a_{ij})$, then the minor of a_{ij} is $\det M_{ij}$

The **cofactor** of an entry a_{ij} is the number $(-1)^{i+j}\det M_{ij}$.

Therefore, the determinant of a square matrix equals the sum of the products of the entries on a row (or column) by their corresponding cofactors.

If the determinant of a square matrix is zero, then the matrix is singular; it has no inverse.

If the determinant of a square matrix is not zero, then the matrix is invertible or regular.

Example 1.3.1.

1. Calculate the determinant of the matrix $A = \begin{pmatrix} 3 & 5 & 1 \\ 1 & 2 & 4 \\ 4 & 3 & 2 \end{pmatrix}$ by expanding along:
   a) The first row
   b) The second column

Solution:

a) $\begin{vmatrix} 3 & 5 & 1 \\ 1 & 2 & 4 \\ 4 & 3 & 2 \end{vmatrix} = 3\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} - 5\begin{vmatrix} 1 & 4 \\ 4 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = 3(-8) - 5(-14) + (-5) = 41$

b) $\begin{vmatrix} 3 & 5 & 1 \\ 1 & 2 & 4 \\ 4 & 3 & 2 \end{vmatrix} = -5\begin{vmatrix} 1 & 4 \\ 4 & 2 \end{vmatrix} + 2\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} = -5(-14) + 2(2) - 3(11) = 41$

2. Expanding along which row or column will make the calculation of the determinant

$\begin{vmatrix} 7 & 23 & 1 \\ 0 & -7 & 0 \\ 13 & -15 & 2 \end{vmatrix}$ easier. Write down the value of the determinant.

Solution:

The expansion along the second row will make the calculation of the determinant easier.

In fact, $\begin{vmatrix} 7 & 23 & 1 \\ 0 & -7 & 0 \\ 13 & -15 & 2 \end{vmatrix} = -7\begin{vmatrix} 7 & 1 \\ 13 & 2 \end{vmatrix} = -7(14-13) = -7$

Application activity 1.3.1

1. Differentiate between $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

2. Calculate the determinant of the matrix $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ by:

   Sarrus' method, (b) By expansion by cofactors

3. State whether the matrix X is singular or not if:

a) $X = \begin{pmatrix} 5 & 0 \\ -3 & 9 \end{pmatrix}$ b) $X = \begin{pmatrix} 4k & -20k \\ -k & 5k \end{pmatrix}$, where k is a constant real number.

1.3.2. Properties of determinants

Learning Activity 1.3.2

1. Without calculation, predict the value of the determinant of each of the following matrices:

a) $A=\begin{pmatrix}1 & 3\\ 2 & 6\end{pmatrix}$ b) $B=\begin{pmatrix}12 & 0 & 351\\ \sqrt{3} & 0 & 16\\ -\frac{5}{7} & 0 & -9\end{pmatrix}$

2. Consider the matrices $A=\begin{pmatrix}-1 & 2\\ 3 & -5\end{pmatrix}$ and $B=\begin{pmatrix}3 & -5\\ -1 & 2\end{pmatrix}$.;

a) How are rows 1 and 2 of matrix B obtained from the rows of matrix A?

b) Calculate $\begin{vmatrix}-1 & 2\\ 3 & -5\end{vmatrix}$ and $\begin{vmatrix}3 & -5\\ -1 & 2\end{vmatrix}$

c) What do you conclude?

CONTENT SUMMARY

A square matrix can be changed into simpler form before calculating its determinant through properties including the following:

1. If all the entries of a row or column of a square matrix are zeros, then the determinant of the matrix is zero.

2. If all the entries of a row (or column) of a square matrix are multiplied by a real number k, then the determinant of the matrix is multiplied by k. Thus, $\begin{vmatrix}a & a' & a''\\ kb & kb' & kb''\\ c & c' & c''\end{vmatrix}=k\begin{vmatrix}a & a' & a''\\ b & b' & b''\\ c & c' & c''\end{vmatrix}$

3. If two rows or columns of a square matrix are identical or proportional, then the determinant of the matrix is zero.

4. If square matrix B is obtained by interchanging two rows or two columns of square matrix A, then the determinant of B is the opposite of the determinant of A .

5. 5)If a row or column of a square matrix B is obtained by adding or subtracting any nonzero multiple of another row or column of matrix A, the other rows or columns of B being the same as those of A ,then the determinant of matrices A and B remains unchanged. Thus,

$$\begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{vmatrix} = \begin{vmatrix} a+ka' & a' & a'' \\ b+kb' & b' & b'' \\ c+kc' & c' & c'' \end{vmatrix}$$

Example 1.3.2.

1. Calculate the determinant of the matrix $A = \begin{pmatrix} 3 & 5 & 4 \\ 1 & 4 & 2 \\ 2 & 3 & 2 \end{pmatrix}$, then, wi hout **further calculation, find:**

t

a) $\begin{vmatrix} 3 & 4 & 5 \\ 1 & 2 & 4 \\ 2 & 2 & 3 \end{vmatrix}$

b) $\begin{vmatrix} 8 & 5 & 4 \\ 5 & 4 & 2 \\ 5 & 3 & 2 \end{vmatrix}$

Solution:

$$\begin{vmatrix} 3 & 5 & 4 \\ 1 & 4 & 2 \\ 2 & 3 & 2 \end{vmatrix} = 3\begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix} - 5\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} + 4\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = 3(8-6) - 5(2-4) + 4(3-8) = -4$$

a) Column 2 and column 3 of matrix A are interchanged $(C_2 \leftrightarrow C_3)$

Therefore, $\begin{vmatrix} 3 & 4 & 5 \\ 1 & 2 & 4 \\ 2 & 2 & 3 \end{vmatrix} = -(-4) = 4$

b) Column2 is added to column1$(C_1 \rightarrow C_1 + C_2)$

The determinant remains unchanged.

$$\begin{vmatrix} 8 & 5 & 4 \\ 5 & 4 & 2 \\ 5 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 5 & 4 \\ 1 & 4 & 2 \\ 2 & 3 & 2 \end{vmatrix} = -4$$

2. Evaluate

$$\begin{vmatrix} 26 & 31 & -1 \\ 53 & 9 & 42 \\ 0 & 0 & 0 \end{vmatrix}$$

Solution:

$\begin{vmatrix} 26 & 31 & -1 \\ 53 & 9 & 42 \\ 0 & 0 & 0 \end{vmatrix}$ =0, since row 3 consists of zeros only.

Application activity 1.3.2

1. Check if the following statements are True or false:

a) $k\begin{vmatrix} a & c \\ b & d \end{vmatrix} = \begin{vmatrix} ka & kc \\ kb & kd \end{vmatrix}$

b) $k\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} ka & kc \\ kb & kd \end{pmatrix}$

2. Apply the indicated transformations to evaluate $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$

a) $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

b) Write down the value of the determinant

1.4. Finding the inverse and solving simultaneous linear equations

1.4.1. Inverse of a matrix

Learning Activity 1.4.1

Given the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

a) Use augmented matrix and the following elementary row operations

i) $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

ii) $R_1 \to R_1 - R_3$

iii) $R_1 \to R_1 - R_2$

b) Write down A^{-1}

c) Perform the following:

i) Find $\det A$

ii) Obtain matrix C ,where each entry of A is replaced by its cofactor

iii) Obtain matrix (denote it $Adj(A)$) by writing the entries of the first row of C as respective entries of the first column of $Adj(A)$, the entries on the second row of C as the respective entries of the second column of $Adj(A)$, and the entries on the third row of C as the respective entries of the third column of $Adj(A)$

iv)Write down matrix $X = \frac{1}{\det A} Adj(A)$

v) Compare A^{-1} and X

CONTENT SUMMARY

Let A be a square matrix of order 2×2 or 3×3 .Then the inverse of A. Can also be calculated through the following four steps:

1. Find the determinant of A, that is $\det A$;

2. Find the matrix C of cofactors of A: each entry of A is replaced by its cofactor.

3. Find the **adjoint** of matrix A, denoted, $Adj(A)$: the **transpose** of the matrix of cofactors;

4. The inverse of matrix A is $A^{-1} = \frac{1}{\det A} Adj(A)$, where matrix A is regular, or invertible.

The transpose of a matrix A of order $n\times p$ is the matrix denoted A^T whose rows are the columns of A and whose columns are the rows of A.

In particular, if $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is an invertible matrix, then $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

Example 1.4.1.

Find the inverse of each of the following matrices:

$$A=\begin{pmatrix}1 & 3\\ 2 & 8\end{pmatrix} \text{ and } B=\begin{pmatrix}3 & 5 & 1\\ 1 & 2 & 4\\ 4 & 3 & 2\end{pmatrix}$$

Solution:

$$A^{-1}=\frac{1}{2}\begin{pmatrix}8 & -3\\ -2 & 1\end{pmatrix}$$

$$\det B=\begin{vmatrix}3 & 5 & 1\\ 1 & 2 & 4\\ 4 & 3 & 2\end{vmatrix}=3\begin{vmatrix}2 & 4\\ 3 & 2\end{vmatrix}-5\begin{vmatrix}1 & 4\\ 4 & 2\end{vmatrix}+\begin{vmatrix}1 & 2\\ 4 & 3\end{vmatrix}=41$$

Matrix of cofactors:

$$C=\begin{pmatrix}+\begin{vmatrix}2 & 4\\ 3 & 2\end{vmatrix} & -\begin{vmatrix}1 & 4\\ 4 & 2\end{vmatrix} & +\begin{vmatrix}1 & 2\\ 4 & 3\end{vmatrix}\\ -\begin{vmatrix}5 & 1\\ 3 & 2\end{vmatrix} & +\begin{vmatrix}3 & 1\\ 4 & 2\end{vmatrix} & -\begin{vmatrix}3 & 5\\ 4 & 3\end{vmatrix}\\ +\begin{vmatrix}5 & 1\\ 2 & 4\end{vmatrix} & -\begin{vmatrix}3 & 1\\ 1 & 4\end{vmatrix} & +\begin{vmatrix}3 & 5\\ 1 & 2\end{vmatrix}\end{pmatrix}=\begin{pmatrix}-8 & 14 & -5\\ -7 & 2 & 11\\ 18 & -11 & 1\end{pmatrix}$$

Adjoint matrix: $Adj(\mathrm{B})=C^T=\begin{pmatrix}-8 & -7 & 18\\ 14 & 2 & -11\\ -5 & 11 & 1\end{pmatrix}$

The inverse is $B^{-1}=\frac{1}{41}\begin{pmatrix}-8 & -7 & 18\\ 14 & 2 & -11\\ -5 & 11 & 1\end{pmatrix}$

Application activity 1.4.1

1. Find the inverse of matrices:

   a) $A=\begin{pmatrix}20 & 6\\ 5 & 2\end{pmatrix}$ and

   b) $M=\begin{pmatrix}5 & 0 & 2\\ 3 & 4 & 5\\ 2 & 1 & 2\end{pmatrix}$

2) Consider of the following matrices $A=\begin{pmatrix}3 & 7 & 2\\ 1 & 0 & 2\\ 0 & 1 & 1\end{pmatrix}$ and $B=\begin{pmatrix}2 & -2 & 1\\ 3 & 1 & 6\\ -1 & 1 & 1\end{pmatrix}$

Find:

i) A^{-1} and B^{-1}

ii) $\left(A^{-1}\right)^{-1}$

iii) $(10A)^{-1}$

1.4.2. Solving simultaneous linear equations using inverse of a matrix

Learning Activity 1.4.2

A business makes floor tiles and wall tiles.

The table below shows the number of tiles of each type and the labor (in hours) for making the tiles:

	Material(tiles)	**Labor(hours)**
Floor tiles	4	3
Wall tiles	11	1

Given that the total cost for floor tiles is 53 (thousand) FRW and the total cost for wall tiles is 37(thousand) FRW, find the material cost and the labor cost by answering the following questions:

a) Label x the material cost and y the labor cost, and then model the problem by simultaneous linear equations in x and y

b) Express the information in the table above as a matrix A of order 2×2, the total floor tile cost and the total wall tile cost as a matrix B of order 2×1, and the material cost and the labor cost as a matrix X of order 2×1

c) Perform the operation $A.X = B$ and compare it to the simultaneous equations obtained in part a)

d) Find the inverse matrix A^{-1} and the product $A^{-1}.B$

e) Using $X = A^{-1}.B$,find the values of x and y.

CONTENT SUMMARY

The two simultaneous linear equations in two unknowns, x and y,

$\begin{cases} ax+by=c \\ a'x+b'y=c' \end{cases}$ can be arranged, using matrices as, $\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} c \\ c' \end{pmatrix}$

Let $A=\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}, X=\begin{pmatrix} x \\ y \end{pmatrix}$ and $B=\begin{pmatrix} c \\ c' \end{pmatrix}$.

Then $A.X = B$.Multiplying both sides by A^{-1}, we have $A^{-1}.(A.X) = A^{-1}.B$,using the associative property, the inverse property , and the identity property of multiplication of matrices , we have:

$$(A^{-1}.A).X = A^{-1}.B$$
$$\Leftrightarrow I.X = A^{-1}.B$$
$$\Leftrightarrow X = A^{-1}.B$$

In the same way, the three simultaneous linear equations in three unknowns,

x , y and z $\begin{cases} ax+by+cz=d \\ a'x+b'y+c'z=d' \\ a''x+b''y+c''x=d'' \end{cases}$ can be arranged, using matrices as,

$$\begin{pmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix}.\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} d \\ d' \\ d'' \end{pmatrix}$$

Let $A=\begin{pmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix}, X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B=\begin{pmatrix} d \\ d' \\ d'' \end{pmatrix}$.Then $A.X=B$.Solving for X,

$$X = A^{-1}.B$$

Example 1.4.2.

Use matrices to solve the following simultaneous equations:

a) $\begin{cases} 2x-3y=-8 \\ 5x+y=-3 \end{cases}$

b) $\begin{cases} 2x-y+z=-1 \\ x+3y-z=13 \\ x+y+z=3 \end{cases}$

Solution:

a) The system of the simultaneous equations can be expressed as

$$\begin{pmatrix} 2 & -3 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -8 \\ -3 \end{pmatrix}$$

Let $A=\begin{pmatrix} 2 & -3 \\ 5 & 1 \end{pmatrix}; X=\begin{pmatrix} x \\ y \end{pmatrix}$ and $B=\begin{pmatrix} -8 \\ -3 \end{pmatrix}$. The system becomes $A.X=B$;

$$X = A^{-1}.B;$$

$$\det A=\begin{vmatrix} 2 & -3 \\ 5 & 1 \end{vmatrix}=2(1)-5(-3)=17;\ A^{-1}=\frac{1}{17}\begin{pmatrix} 1 & 3 \\ -5 & 2 \end{pmatrix};$$

$X=\frac{1}{17}\begin{pmatrix} 1 & 3 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} -8 \\ -3 \end{pmatrix}=\frac{1}{17}\begin{pmatrix} -17 \\ 34 \end{pmatrix}=\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, that is $x=-1$ and $y=2$

Therefore, the solution set of the simultaneous equations is $S=\{(-1,2)\}$

b) The system of the simultaneous equations can be expressed as

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 13 \\ 3 \end{pmatrix}$$

Let $A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix}; X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} -1 \\ 13 \\ 3 \end{pmatrix}$.

The system becomes $A.X = B$; $X = A^{-1}.B$;

$\det A = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = 8$ (Expansion along the third row).

Matrix of cofactors:

$$C = \begin{pmatrix} +\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 4 & -2 & -2 \\ 2 & 1 & -3 \\ -2 & 3 & 7 \end{pmatrix}$$

The adjoint matrix is $Adj(A) = \begin{pmatrix} 4 & 2 & -2 \\ -2 & 1 & 3 \\ -2 & -3 & 7 \end{pmatrix}$,

he inverse is $A^{-1} = \frac{1}{8}\begin{pmatrix} 4 & 2 & -2 \\ -2 & 1 & 3 \\ -2 & -3 & 7 \end{pmatrix}$;

T

$X = \frac{1}{8}\begin{pmatrix} 4 & 2 & -2 \\ -2 & 1 & 3 \\ -2 & -3 & 7 \end{pmatrix}\begin{pmatrix} -1 \\ 13 \\ 3 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 16 \\ 24 \\ -16 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$, that is $x = 2$, $y = 3$ and $z = -2$

Therefore, the solution set of the simultaneous equations is $S = \{(2,3,-2)\}$

Application activity 1.4.2

Use matrices to solve the following simultaneous equations:

a) $\begin{cases} 4x + 7y = 34 \\ 3x - 2y = 11 \end{cases}$

b) $\begin{cases} x - 2y + z = -2 \\ 3x + y - 2z = 7 \\ x + 3y - z = 2 \end{cases}$

1.4.3. Solving simultaneous linear equations using Cramer's rule

Learning Activity 1.4.3

Consider the simultaneous linear equations $\begin{cases} ax + by = c (1) \\ a'x + b'y = c' (2) \end{cases}$

1. a) Multiply both sides of equation (1) by b' to get equation (3), and multiply both sides of equation (2) by $-b$ to get equation (4)

   b) Perform the addition $(3)+(4)$ to obtain equation (5)

   c) Make x the subject of formula in equation (5), precising the condition for this operation to be valid(possible).Label (6) this equation.

   d) Express the numerator and the denominator of (6) as determinants of matrices of order 2×2

2. a) Multiply both sides of equation (1) by $-a'$ to get equation (3') and multiply both sides of equation (2) by a to get equation (4')

   b) Perform the addition $(3')+(4')$ to obtain equation (5')

c) Make x the subject of formula in equation $(5')$, precising the condition for this operation to be valid(possible).Label $(6')$ this equation.

d) Express the numerator and the denominator of $(6')$ as determinants of matrices of order 2×2

3. Use the formulas you have obtained above to solve the simultaneous equations:

$$\begin{cases} 3x+2y=4 \\ 5x-4y=14 \end{cases}$$

CONTENT SUMMARY

To solve the two simultaneous linear equations in two **unknowns** x **and** y, Cramer's rule requires to go through the following steps:

1. Arrange the equations to get $\begin{cases} ax+by=c \\ a'x+b'y=c' \end{cases}$ and calculate the **principal determinant** $D=\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}=ab'-a'b$

If $D=0$, then the system has no solution or infinitely many solutions; the system is not a Cramer's system.

If $D\neq 0$, then the system is a Cramer's system and has unique solution, proceed to the next step:

2. Write down and calculate: $D_x=\begin{vmatrix} c & b \\ c' & b' \end{vmatrix}=cb'-c'b$ and $D_y=\begin{vmatrix} a & c \\ a' & c' \end{vmatrix}=ac'-a'c$

3. Write down and calculate $x=\dfrac{D_x}{D}$ and $y=\dfrac{D_y}{D}$; the solution set of the simultaneous equations is $S=\left\{\left(\dfrac{D_x}{D},\dfrac{D_y}{D}\right)\right\}$

In the same way, for the three simultaneous linear equations in three unknowns, x , y and z , $\begin{cases} ax+by+cz=d \\ a'x+b'y+c'z=d' \\ a''x+b''y+c''x=d'' \end{cases}$

The principal determinant is $D=\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}$

If $D=0$, then the system is not a Cramer's system, it may have zero solution or infinitely many solutions.

If $D\neq 0$, then the system is a Cramer's system and has unique solution; the solution set is $S=\left\{\left(\frac{D_x}{D},\frac{D_y}{D},\frac{D_z}{D}\right)\right\}$, where

$$D_x=\begin{vmatrix} d & b & c \\ d' & b' & c' \\ d'' & b'' & c'' \end{vmatrix},\quad D_y=\begin{vmatrix} a & d & c \\ a' & d' & c' \\ a'' & d'' & c'' \end{vmatrix},\quad D_z=\begin{vmatrix} a & b & d \\ a' & b' & d' \\ a'' & b'' & d'' \end{vmatrix}$$

Example 1.4.3.

Use Cramer's rule to solve the simultaneous equations: $\begin{cases} 3x+2y-2z=4 \\ x+3y+z=7 \\ 2x+y-z=11 \end{cases}$

Solution: $D=\begin{vmatrix} 3 & 2 & -2 \\ 1 & 3 & 1 \\ 2 & 1 & -1 \end{vmatrix}=3\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix}-2\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}-2\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}=4$

$$D_x=\begin{vmatrix}4&2&-2\\7&3&1\\11&1&-1\end{vmatrix}=4\begin{vmatrix}3&1\\1&-1\end{vmatrix}-2\begin{vmatrix}7&1\\11&-1\end{vmatrix}-2\begin{vmatrix}7&3\\11&1\end{vmatrix}=72$$

$$D_y=\begin{vmatrix}3&4&-2\\1&7&1\\2&11&-1\end{vmatrix}=3\begin{vmatrix}7&1\\11&-1\end{vmatrix}-\begin{vmatrix}4&-2\\11&-1\end{vmatrix}+2\begin{vmatrix}4&-2\\7&1\end{vmatrix}=-36$$

$$D_z=\begin{vmatrix}3&2&4\\1&3&7\\2&1&11\end{vmatrix}=3\begin{vmatrix}3&7\\1&11\end{vmatrix}-\begin{vmatrix}2&4\\1&11\end{vmatrix}+2\begin{vmatrix}2&4\\3&7\end{vmatrix}=64$$

$$x=\frac{D_x}{D}=\frac{72}{4}=18$$

$$y=\frac{D_y}{D}=\frac{-36}{4}=-9$$

$$z=\frac{D_z}{D}=\frac{64}{4}=16$$

The solution set is $S=\{(18,-9,16)\}$

Application activity 1.4.3

Use Cramer's rule to solve the following simultaneous equations:

a) $\begin{cases}4x+7y=34\\3x-2y=11\end{cases}$

b) $\begin{cases}x-2y+z=-2\\3x+y-2z=7\\x+3y-z=2\end{cases}$

End of unit assessment 1

1. Write down the order of each of the following matrices:

a) $A = \begin{pmatrix} -3 & 4 & 2 \\ 1 & 5 & -1 \end{pmatrix}$ b) $B = \begin{pmatrix} 4 & -1 \end{pmatrix}$

2. Given that matrices A and B are equal, find the values of the letters:

a) $A = \begin{pmatrix} 3m+2 & 2 \\ 2k+1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} m-3 & 2 \\ 5 & 6 \end{pmatrix}$

b) $A = \begin{pmatrix} x-1 & -3 \\ x+4z & 3y+4 \end{pmatrix}$ and $B = \begin{pmatrix} 6x+5 & 2 \\ 1 & 1 \end{pmatrix}$

3. Perform each of the following operations:

a) $3\begin{pmatrix} -2 & 3 \\ 1 & 2 \end{pmatrix} - 2\begin{pmatrix} 1 & 4 \\ -1 & -1 \end{pmatrix}$

b) $\begin{pmatrix} -1 & 3 & 1 \\ 2 & 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 1 & -1 \\ -1 & 2 \end{pmatrix}$

4. Invertible 2×2 matrices A, B and X are such that $4A-5BX = B$

a) Make X the subject of the formula

b) Find X if $A = 2B$

5. a) Given that $A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix}$, and $B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, find the products

$A.B$ and $B.A$

c) Calculate the product $\begin{pmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix} . \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 2 & 3 \end{pmatrix}$

6. Find the inverse matrix of each of the following matrices:

a) $A = \begin{pmatrix} -1 & 4 \\ 2 & 3 \end{pmatrix}$ b) $M = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix}$

7. a) The transpose of matrix $M = \begin{pmatrix} 1 & 2 & x^2 \\ 4 & 1 & 0 \\ 1 & x+3 & 8 \end{pmatrix}$ is $M^T = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 1 & 1 \\ 4 & 0 & 8 \end{pmatrix}$.

Find the value of x

b) Use $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $B = \begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix}$ to derive the formulas for :

i) $(A^T)^T$: the transpose of the transpose of a matrix

ii)) $(A.B)^T$; the transpose of a product

8. Evaluate the following:

a) a) $\begin{vmatrix} 12 & 6 \\ 5 & 4 \end{vmatrix}$ b) $\begin{vmatrix} 3 & 2 & 1 \\ 0 & 2 & -5 \\ -2 & 1 & 4 \end{vmatrix}$

9. a) In the calculation of $\begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix}$, along which row or column is it advantageous to expand by cofactors? Write down the value of the determinant

b) i) Evaluate $\begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix}$, using expansion by cofactors along row 2.

ii) Name the transformation applied on the determinant $\begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix}$ to obtain the determinant $\begin{vmatrix} 2 & 1 & 7 \\ 1 & 2 & 4 \\ 3 & 5 & 17 \end{vmatrix}$.

iii) write down the value of $\begin{vmatrix} 2 & 1 & 7 \\ 1 & 2 & 4 \\ 3 & 5 & 17 \end{vmatrix}$

10.

a) Use inverse matrix to solve the simultaneous linear equations:

$$\begin{cases} x + y - z = 0 \\ x + 2y + 3z = 14 \\ 2x + y + 4z = 16 \end{cases}$$

b) Use Cramer's rule to solve the simultaneous equations:

$$\begin{cases} x + y + z = 6 \\ 2x + y - z = 1 \\ 3x + 2y + z = 10 \end{cases}$$

Unit 2 Differentiation/Derivatives

Key Unit competence: Solve Economical, Production, and Financial related problems using derivatives.

Introductory activity

Consider functions $y = 3x - 2$ (1), and $y = x^2 + 1$ (2)

a) complete the following table for each of the two functions:

x_1	x_2	$\Delta x = x_2 - x_1$	y_1	y_2	$\Delta y = y_2 - y_1$	$\frac{\Delta y}{\Delta x}$
1	2					
1	1.5					
1	1.1					
...	...	...	...	...	...	...

b) How is the quantity $\frac{\Delta y}{\Delta x}$ called?

c) Compare $\frac{\Delta y}{\Delta x}$ for function (1) and for function (2)

d) Predict a formula for $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

2.1 Differentiation from first principles

2.1.1. Average rate of change of a function

Learning Activity 2.1.1

Suppose that the profit by selling x units of an item is modeled by the equation, $P(x) = 4x^2 - 5x + 3$, and x assumes values 2 and 5, respectively. Find:

a) The change in x

b) The values of P for $x = 2$ and $x = 5$, respectively. Hence, find the change in P

c) Find the ratio of the change in P to the change in x

d) Give a word with the same meaning as ratio

CONTENT SUMMARY

The **average rate of change** of function , $y = f(x)$ as the independent variable x assumes values from x_1 to x_2, is the quantity defined by $\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

Example 2.1.1.

The cost (in thousand FRW) of producing a certain commodity is modeled by the function $C(x) = 40 + \sqrt{x}$. Find the average rate of change of the cost when the production level changes from $x = 64$ to $x = 100$.

Solution:

The change in the independent variable is $\Delta x = 100 - 64 = 36$; $C(100) = 40 + \sqrt{100} = 50$ and $C(64) = 40 + \sqrt{64} = 48$; The change in the dependent variable is $\Delta C = C(100) - C(64) = 50 - 48 = 2$, The average rate of change in the cost is $\frac{\Delta C}{\Delta x} = \frac{2}{36} = \frac{1}{18}$

Application activity 2.1.1

1. In a Forex bureau, for 100 units of currency y, you obtain $107,500$ units of currency x. Assuming that the two currencies are related by a linear equation, find the average rate of change of currency y, with respect to currency x.

2. Find the average rate of change of function $f(x) = x^2 - \frac{1}{x}$ as x assumes values from 2 to 4.

2.1.2. Instantaneous rate of change of a function

Learning Activity 2.1.2

Consider function $y = f(x) = x^2 + 1$, and the changes in x from $x_0 = 2$ to x_1, where x_1 assumes consecutively values $x_1 = 2.1; x_1 = 2.01; x_1 = 2.001; \ldots$

a) Complete the following table:

x_1	2.1	2.01	2.001	etc	$x_1 \to \ldots$
$\Delta x = x_1 - x_0$				etc	$\Delta x \to \ldots$
$\Delta y = y_1 - y_0$				etc	/////////////
$\frac{\Delta y}{\Delta x}$				etc	$\frac{\Delta y}{\Delta x} \to \ldots$

b) How is the statement "If $\Delta x \to a$ then $\frac{\Delta y}{\Delta x} \to b$" written in terms of limits?

c) Find, in terms of x, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ for $y = 3x^2$ at any value of x.

CONTENT SUMMARY

The instantaneous rate of change of funct$_{\text{i}}$on $y = f(x)$ at $x = x_0$ is the value

f $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ $_{\text{f}}$or $x = x_0$.

o

By finding the instantaneous rate of change of function $y = f(x)$ by using limit, we say that we are differentiating function $y = f(x)$ from **first principles.**

Equivalently, the instantaneous rate of change of function $y = f(x)$ at x_0 is the number, $\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$, provided it exists.

This number is called the **derivative** of function $y = f(x)$ at x_0; it is denoted by $f'(x_o)$. More generally, the derivative of function $y = f(x)$ is denoted by $f'(x)$ or y' or $\frac{dy}{dx}$

Example 2.1.2.

Differentiate function $y = f(x) = \frac{1}{x}$ from first principles.

Solution:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{1}{x + \Delta x} - \frac{1}{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-1}{x(x + \Delta x)}$$

$$= \frac{-1}{x^2}$$

Application activity 2.1.2

1. Find the instantaneous rate of change of the total cost $C(Q)=Q^2+7Q+23$, with respect to the number Q of units sold

2. Differentiate, from first principles, $f(x)=\sqrt{x}-3$

2.2. Rules for differentiation

2.2.1. Differentiation of polynomial functions

Learning Activity 2.2.1

Consider the following functions:

$$f(x)=3x^2-2x+\sqrt{x}+1;\ g(x)=\frac{1}{3}x^4+5x-\sqrt{7}\ \text{and}\ h(x)=4x^{-3}+7x-5$$

a) Which of the three functions is a polynomial function?

b) Describe the quantities and the operations involved in a polynomial

c) Obtain, from first principles, the derivative of each of the following:

i) A constant function C

ii) A sum of two functions $u(x)$ and $v(x)$

iii) A product of a real number k by a function $u(x)$

iv) $x; x^2; x^3$, and then predict the derivative of x^n, where x is the variable and n a positive integer.

d) Write down the derivative of $P(x)=a_0+a_1x+a_2x^2+...+a_nx^n$, where $a_0, a_1, a_2, ..., a_n$ are constant real numbers, $0, 1, 2, ..., n$ are positive integers, and x is the variable.

CONTENT SUMMARY

A polynomial function in one variable x is a function of the type $P(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ .Thus, a polynomial consists of a variable raised to different positive integral powers, powers are multiplied by constants and the products are added, or subtracted. It can be shown, from first principles, that:

1. $\frac{d}{dx}(C) = 0$: the derivative of a constant is 0;
2. $\frac{d}{dx}(u+v) = \frac{d}{dx}(u) + \frac{d}{dx}(v)$: the derivative of a sum equals the sum of derivatives of the terms, where u and v are functions of variable x;
3. $\frac{d}{dx}(ku) = k\frac{d}{dx}(u)$, where k is a constant and u is a function of variable x: the derivative of the product of a constant real number by a function equals the product of the real number by the derivative of the function;
4. $\frac{d}{dx}(x^n) = nx^{n-1}$; the derivative of the nth power of the variable equals the product of the exponent by the (n-1)th power of the variable. From the properties above, it follows that:

$$\frac{d}{dx}(a_0 + a_1x + a_2x^2 + ... + a_nx^n) = a_1 + 2a_2x + ... + na_nx^{n-1}$$

Example 2.2.1.

Find the derivative of each of the following polynomials:

a) $f(x) = 2x^3 - 5x^2 + 11x - 3$

b) $g(x) = -\frac{1}{3}x^3 + \frac{3}{4}x^2 - 25$

Solution:

a) $f'(x) = 6x^2 - 10x + 11$

b) $g'(x) = -x^2 + \frac{3}{2}x$

Application activity 2.2.1

Find the derivative $\frac{dz}{dt}$ or $\frac{dP}{dQ}$ of:

a) $z = 2 - 8t^3 + 3t^5$

b) $P = 5Q^4 + 3Q - 7$

2.2.2. Differentiation of product functions

Learning Activity 2.2.2

Let $u = -3x^2 + 2$ and $v = 4x - 9$:

a) i) Expand the product uv

ii) Calculate $\frac{d}{dx}(uv)$

b) i) Find $\frac{du}{dx}$ and $\frac{dv}{dx}$

ii) Calculate: $v\frac{du}{dx} + u\frac{dv}{dx}$

c) Compare $\frac{d}{dx}(uv)$ and $v\frac{du}{dx} + u\frac{dv}{dx}$

CONTENT SUMMARY

To find the derivative of the product of two functions u and v of independent variable x, one can proceed as follows: Either expand the product, and then find the derivative of the expansion, Or, calculate the derivative of each factor, and then obtain the products $v\frac{du}{dx}$ and $u\frac{dv}{dx}$, and finally, consider the sum $v\frac{du}{dx} + u\frac{dv}{dx}$, that is, $\frac{d}{dx}(uv) = v\frac{d}{dx}(u) + u\frac{d}{dx}(v)$

It is the matter of choosing the more convenient method.

Example 2.2.2.

Find the derivative of each of the following products:

a) $f(x) = (2x^3 - 5x^2)(11x - 3)$

b) $g(x) = -\frac{1}{3}x^3(\frac{3}{4}x^2 - 25)$

Solution:

Let $u = 2x^3 - 5x^2$ and $v = 11x - 3$. Then, $\frac{d}{dx}(u) = 6x^2 - 10x$ and $\frac{d}{dx}(v) = 11$

Substituting these values in the product rule formula,

$$\frac{d}{dx}(uv) = (11x - 3)(6x^2 - 10x) + 11(2x^3 - 5x^2)$$

$$= 88x^3 - 183x^2 + 30x$$

Let $u = -\frac{1}{3}x^3$ and $v = \frac{3}{4}x^2 - 25$, Then $\frac{d}{dx}(u) = -x^2$ and $\frac{d}{dx}(v) = \frac{3}{2}x$

Substituting these values in the product rule formula,

$$\frac{d}{dx}(uv) = (\frac{3}{4}x^2 - 25)(-x^2) + (-\frac{1}{3}x^3)(\frac{3}{2}x)$$

$$= -\frac{5}{4}x^4 + 25x^2$$

Application activity 2.2.2

Find the derivative $\frac{dz}{dt}$ or $\frac{dP}{dQ}$ of:

a) $z = (2 - 8t^3)(4 + 3t^5)$

b) $P = 5Q^4(3Q - 7)$

2.2.3. Differentiation of power functions

Learning Activity 2.2.3

Let $y=(2x+1)^3$

a) Expand y and then calculate $\frac{dy}{dx}$

b) i)Find $\frac{d}{dx}(2x+1)$

ii) Calculate $3(2x+1)^2\frac{d}{dx}(2x+1)$

iii) Describe how to obtain $3(2x+1)^2$ from $(2x+1)^3$

c) Predict a rule for finding $\frac{d}{dx}(u^n)$, where u is function of variable x

CONTENT SUMMARY

Let $y=u^n$, where u is function of variable x and n a rational number. It can be shown that $\frac{d}{dx}(u^n)=nu^{n-1}\frac{du}{dx}$

To find the derivative of the nth power of function u of independent variable x, one can proceed as follows: Either expand the power, and then find the derivative of the expansion,

Or, calculate the derivative of the base, and then apply the formula

$$\frac{d}{dx}(u^n)=nu^{n-1}\frac{du}{dx}$$

It is the matter of choosing the more convenient method.

Example 2.2.3

Find the derivative of each of the following powers:

a) $f(x)=(2x^3+3x^2-5x+1)^6$

b) $g(x)=\sqrt{3x+2}$

Solution:

a) $f'(x) = 6(2x^3 + 3x^2 - 5x + 1)^5 (6x^2 + 6x - 5)$

b) We have: $\sqrt{3x+2} = (3x+2)^{\frac{1}{2}}$, $\frac{d}{dx}(\sqrt{3x+2}) = \frac{1}{2}(3x+2)^{\frac{1}{2}-1}\frac{d}{dx}(3x+2)$

$$= \frac{1}{2}(3x+2)^{-\frac{1}{2}}(3)$$

$$= \frac{3}{2\sqrt{3x+2}}$$

Application activity 2.2.3

Find the derivative of:

a) $y = (7x+8)^2$

b) $y = (4x-5)^3$

c) $P = 5Q^4(3Q-7)$

2.2.4. Differentiation of the composite function (The chain rule)

Learning Activity 2.2.4

Let $y = (2x+1)^3$

a) Use the power rule to calculate $\frac{dy}{dx}$

b) i) Determine two functions $u(x)$ and $v(u)$ such that $y = v \circ u$, that is $y = v[u(x)]$

ii) Find $\frac{du}{dx}$ and $\frac{dv}{du} = \frac{dy}{du}$

iii) Calculate $\frac{dy}{du} \cdot \frac{du}{dx}$

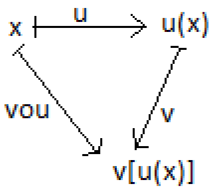
c) Compare $\frac{dy}{dx}$ and $\frac{dy}{du}.\frac{du}{dx}$

iii) Describe how to obtain $3(2x+1)^2$ from $(2x+1)^3$

d) Predict a rule for finding $\frac{d}{dx}(u^n)$, where u is function of variable x

CONTENT SUMMARY

Consider the following diagram:



If function $y = f(x)$ can be expressed as $y = v[u(x)]$, where $u(x)$ and $v(u)$ are functions to determine, **then, it** can be shown that: $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$. This formula is known as the chain rule.

Example 2.2.4.

Use the chain rule to find the derivative of:

a) $y = (4x+5)^6$

b) $y = \sqrt{3x+2}$

Solution:

a) Let $u = 4x+5$.Then $\frac{du}{dx} = 4$, the function becomes $y = u^6$. We have

$\frac{dy}{du} = 6u^5 = 6(4x+5)^5$,

From the chain rule, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 6(4x+5)^5(4) = 24(4x+5)^5$

b) Let $u = 3x+2$. Then, $\frac{du}{dx} = 3$, the function becomes $y = u^{\frac{1}{2}}$ We have

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{3x+2}},$$

From the chain rule, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{1}{2\sqrt{3x+2}}(3) = \frac{3}{2\sqrt{3x+2}}$

Application activity 2.2.4

Use the chain rule to find the derivative of:

a) $y = (7x+8)^2$

b) $y = (4x-5)^3$

2.2.5. Differentiation of quotient functions

Learning Activity 2.2.5

Consider the function $y = \frac{3x^2}{2x+1}$

a) i) Express y in the form $y = u.v^{-1}$, stating the value of u and the value of v

ii) Use the product rule to find the derivative of $y = u.v^{-1}$ and, express your answer with positive exponents, in the form of a fraction $\frac{N(x)}{D(x)}$

b) Calculate $\frac{(3x+1)\frac{d}{dx}(3x^2)-(3x^2)\frac{d}{dx}(3x+1)}{(3x+1)^2}$

c) Compare $\frac{d}{dx}(\frac{u}{v})$ and $\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

d) Predict a rule for finding $\frac{d}{dx}(\frac{u}{v})$, where u and v are functions of variable x

CONTENT SUMMARY

It can be shown that, if u and v are functions of variable x, and $v(x) \neq 0$,

Then $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, that is, $\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$

Example 2.2.5.

Use the quotient rule to find the derivative of:

a) $y = \frac{x^2 - 3x + 2}{2x + 1}$

b) $y = \frac{2 - x}{\sqrt{x}}$

Solution:

Let $u = x^2 - 3x + 2$ and $v = 2x + 1$.

Then $\frac{du}{dx} = 2x - 3$ and $\frac{dv}{dx} = 2$.

We have $\frac{dy}{dx} = \frac{(2x+1)(2x-3) - 2(x^2 - 3x + 2)}{(2x+1)^2}$

$$= \frac{2x^2 + 3x - 7}{(2x+1)^2}$$

Let $u = 2 - x$ and $v = \sqrt{x}$.

Then $\frac{du}{dx} = -1$ and $\frac{dv}{dx} = \frac{1}{2\sqrt{x}}$.

We have: $\frac{dy}{dx} = \frac{-\sqrt{x} - \frac{2-x}{2\sqrt{x}}}{x} = \frac{-x-2}{2x\sqrt{x}}$

Application activity 2.2.4

Use the quotient rule to find the derivative of:

a) $y = \frac{5x-6}{7x-4}$

b) $y = \frac{3x^4}{2+5x}$

2.2.6. Differentiation of logarithmic functions

Learning Activity 2.2.6

Consider function $f(x) = \ln x$

a) Using a calculator, complete the following table:

x_0 \ Δx	1	2	3	...
0.1	$\frac{\Delta y}{\Delta x} = \frac{\ln(1+0.1)-\ln 1}{0.1} =$	$\frac{\Delta y}{\Delta x} = \frac{\ln(2+0.1)-\ln 2}{0.1} =$	$\frac{\Delta y}{\Delta x} = \frac{\ln(3+0.1)-\ln 3}{0.1} =$	
0.01	$\frac{\Delta y}{\Delta x} = \frac{\ln(1+0.01)-\ln 1}{0.01} =$	$\frac{\Delta y}{\Delta x} = \frac{\ln(2+0.01)-\ln 2}{0.01} =$	$\frac{\Delta y}{\Delta x} = \frac{\ln(3+0.01)-\ln 3}{0.01} =$	
0.001	$\frac{\Delta y}{\Delta x} = \frac{\ln(1+0.001)-\ln 1}{0.001} =$	$\frac{\Delta y}{\Delta x} = \frac{\ln(2+0.001)-\ln 2}{0.001} =$	$\frac{\Delta y}{\Delta x} = \frac{\ln(3+0.001)-\ln 3}{0.001} =$	
...				

b) Use the table above to write the values of $f'(1); f'(2); f'(3);\dots f'(x)$

CONTENT SUMMARY

The derivative of function $y = f(x) = \ln x$ is $f'(x) = (\ln x)' = \frac{1}{x}$, that is $\frac{d}{dx}(\ln x) = \frac{1}{x}$, where $x > 0$. More generally, if $y = \ln[u(x)]$, where $u(x)$ is function of variable x, then from the chain rule, $\frac{d}{dx}[\ln u(x)] = \frac{1}{u}.\frac{du}{dx}$, that is $[\ln u(x)]' = \frac{u'}{u}$.

In the same way, if $\log_a x = \frac{1}{\ln a}.\ln x$, then $(\log_a x)' = \frac{1}{\ln a}(\ln x)' = \frac{1}{x \ln a}$, where $a > 0$ and $a \neq 1$, and $x > 0$.

More generally,

If $y = \log_a u(x)$, where $u(x)$ is function of variable x, then $[\log_a u(x)]' = \frac{u'}{u \ln a}$

Example 2.2.6.

Find the derivative of:

a) $y = \ln(x^2 + 3)$

b) $y = \log(1 - x^2)$

Solution:

a) $y' = \frac{2x}{x^2 + 3}$

b) $y' = \frac{-2x}{(1 - x^2)\ln 10}$

Application activity 2.2.6

Find the derivative of:

a) $y = \ln 4x^3$

b) $y = \log(5x + 6)$

2.2.7. Differentiation of exponential functions

Learning Activity 2.2.7

1. Consider the equality $y = a^x$, where $a > 0$ and $a \neq 1$
   a) Make x the subject of the formula
   b) Differentiate, with respect to x, both side of the equation obtained in part (a)
   c) Make y' the subject of the formula in part (b)
   d) Complete:
      i) $(a^x)' = \ldots$
      ii) in particular, $(e^x)' = \ldots$
2. Use the chain rule to obtain the formula for:
   i) $(a^u)'$
   ii) $(e^u)'$, where $u(x)$ is function of variable x

CONTENT SUMMARY

The derivative of function $y = f(x) = e^x$ is $f'(x) = (e^x)' = e^x$, that is $\frac{d}{dx}(e^x) = e^x$. More generally, if $y = e^{u(x)}$, where $u(x)$ is function of variable x, then from the chain rule, $\frac{d}{dx}\left[e^{u(x)}\right] = e^{u(x)}\frac{du}{dx}$, that is $\left[e^{u(x)}\right]' = u'e^u$. In the same way, if a^x, then $\left(a^x\right)' = a^x \ln a$, where $a > 0$ and $a \neq 1$, and $x > 0$. More generally, If $y = a^{u(x)}$ where $u(x)$ is function of variable x, then $\left[a^{u(x)}\right]' = u'a^u \ln a$

Example 2.2.7.

Find the derivative of:

a) $y = e^{x^2+3}$

b) $y = 10^{1-x^2}$

Solution:

a) $y' = 2xe^{x^2+3}$

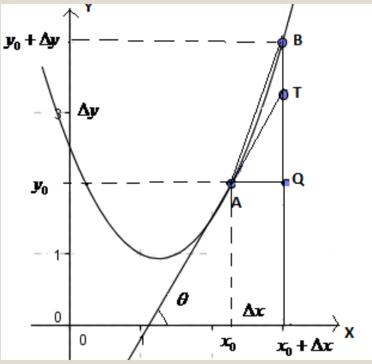
b) $y' = (-2x)(10^{1-x^2})\ln 10$

2.3. Some applications of derivatives in Mathematics

2.3.1. Equation of the tangent to the graph of a function at a point.

Learning Activity 2.3.1

Let $y = f(x)$ beanumericalfunction,and $A\left(x_0, f(x_0)\right)$, $B\left(x_0 + \Delta x, f(x_0 + \Delta x)\right)$ two points on the graph of $y = f(x)$, see the diagram below:



a) Write down the gradient of the straight line through points A and B

b) As point B moves along the curve towards point A:

i) The change in x, that is Δx approaches which value?

ii) The gradient of the straight line through points A and B approaches which value? (Express your answer in terms of the derivative of function f

iii) How becomes the position of the line through points A and B with respect to the graph of function $y = f(x)$

c) Write down the equation of the straight line through points A and T.

CONTENT SUMMARY

The derivative of function $y = f(x)$ at x_0 is the gradient m of the geometric tangent to the graph of $y = f(x)$ at point A, that is, if m is the gradient of the tangent then, $m = f'(x_0)$.

Therefore, the equation of the tangent to the graph of function $y = f(x)$ at x_0 is:

$y - f(x_0) = f'(x_0)(x - x_0)$, since the tangent passes through point $A(x_0, f(x_0))$

Example 2.3.1.

Find the equation of the tangent to the graph of function $f(x)$ at x_0:

a) $f(x) = -2x^2 + 80x$; $x_0 = 10$

b) $f(x) = 200 - 40\ln(x+1)$; $x_0 = 19$

Solution:

a) $f(x_0) = f(10) = -2(10)^2 + 80(10) = 600$; $f'(x) = -4x + 80$; $f'(10) = -4(10) + 80 = 40$;

The equation of the tangent is: $y - 600 = 40(x - 10)$; or, equivalently, $y = 40x + 200$

b) $f(x_0) = f(19) = 200 - 40\ln(19+1) = 200 - 40\ln 20 \approx 80.17$;

$f'(x) = \dfrac{-40}{x+1}$; $f'(19) = \dfrac{-40}{19+1} = -2$;

The equation of the tangent is: $y - 80.17 = -2(x - 19)$; or, equivalently, $y = -2x + 98.17$

Application activity 2.3.1

Find the equation of the tangent to the graph of function $f(x)$ at x_0:

a) $f(x) = 15xe^{-\frac{x}{3}}$; $x_0 = 6$ (leave your answer in terms of e)

b) $f(x) = \sqrt{10 - \frac{1}{2}x}$; x_0 (Leave your answer in terms of surds)

2.3.2. Hospital's rule.

Learning Activity 2.3.2

Let numerical functions $f(x)$ and $g(x)$ be such that $\lim_{x\to x_0}\frac{f(x)}{g(x)}=\frac{f(x_0)}{g(x_0)}=\frac{0}{0}$ or $\frac{\infty}{\infty}$

a) Determine the equations of the tangents to the graphs of functions $f(x)$ and $g(x)$ at x_0, in the form $y=ax+b$ and $y=cx+d$

b) Calculate $\lim_{x\to x_0}\frac{f(x)}{g(x)}$, substituting $f(x)$, and $g(x)$, respectively by $ax+b$ and $cx+d$, obtained from (a) and simplify.

c) Express $\lim_{x\to x_0}\frac{f(x)}{g(x)}$ in terms of a limit involving $f'(x_0)$ and $g'(x_0)$

CONTENT SUMMARY

If num ric l functions $f(x)$ and $g(x)$ are such that $\lim_{x\to x_0}\frac{f(x)}{g(x)}=\frac{f(x_0)}{g(x_0)}=\frac{0}{0}$ or $\frac{\infty}{\infty}$ e a, then to remove the indetermin**ation, we proceed** as follows, t**hrough**

Hospital's rule:

1. Differentiate separately, the numerator and the denominator, to get $f'(x)$ and $g'(x)$;

2. Calculate $\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=\frac{f'(x_0)}{g'(x_0)}$

3. Then $\lim_{x\to x_0}\frac{f(x)}{g(x)}=\frac{f'(x_0)}{g'(x_0)}$

Note that: - the process can be repeated if necessary;

– Hospital's rule is used only if we have indetermination $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Hospital's rule**, is not** the quotient rule for differentiation, that is

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} \neq \lim_{x\to x_0}\left(\frac{f(x)}{g(x)}\right)'$$

– Before applying Hospital's rule, ensure that you have indetermination

$$\frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

Example 2.3.2.

Evaluate the following limits:

$$\lim_{x\to e}\frac{1-\ln x}{x-e}$$

$$\lim_{x\to 0}\frac{e^{3x}-1}{x^2}$$

Solution:

a) $\lim_{x\to e}\frac{1-\ln x}{x-e} = \frac{1-\ln e}{e-e} = \frac{1-1}{e-e} = \frac{0}{0}$**:** indeterminate case

Applying Hospital's rule,

We have: $\lim_{x\to e}\frac{1-\ln x}{x-e} = \lim_{x\to e}\frac{(1-\ln x)'}{(x-e)'} = \lim_{x\to e}\frac{-\frac{1}{x}}{1} = -\lim_{x\to e}\frac{1}{x} = -\frac{1}{e}$

b) $\lim_{x\to 0}\frac{e^{3x}-3x-1}{x^2} = \frac{e^{3(0)}-3(0)-1}{0^2} = \frac{1-1}{0^2} = \frac{0}{0}$**:** indeterminate case

Applying Hospital's rule,

We have: $\lim_{x\to 0}\frac{e^{3x}-3x-1}{x^2} = \lim_{x\to 0}\frac{(e^{3x}-3x-1)'}{(x^2)'} = \lim_{x\to 0}\frac{3e^{3x}-3}{2x} = \frac{0}{0}$**:** Indeterminate case**.**

Repeating the process: $\lim_{x\to 0}\frac{3e^{3x}-3}{2x} = \lim_{x\to 0}\frac{(3e^{3x}-3)'}{(2x)'} = \lim_{x\to 0}\frac{9e^{3x}}{2} = \frac{9}{2}$

Application activity 2.3.2

Evaluate the following limits:

a) $\lim_{x\to 1}\frac{\sqrt{x}-1}{(x-1)e^{x}}$

b) $\lim_{x\to 1}\frac{e^{2x-2}-1}{\ln(5x-4)}$

End of unit assessment 2

1. For function $f(x)=\ln x$, find:

   a) The average rate of change of $f(x)$ as x assumes values from 1 to 1.2

   b) The instantaneous rate of change at $x_0 = 1$

2. Find, from first principles, the derivative of:

   a) $f(x)=3x^2$

   b) $f(x)=\frac{-5}{x}$

3. Find the derivative of:

   a) $y=30x-0.5x^2$

   b) $y=\frac{1}{x}+2\sqrt{x}-3$

4. Differentiate, with respect to t:

   a) $s=3t^4(2t-5)$

   b) $s=(t^7-4)(t^5+11)$

5. Find $\frac{dy}{dx}$,using the chain rule, if:

   a) $y=\sqrt{x^2-x+2}$

   b) $y=(3x^4+7)^6$

6. Find $\frac{dQ}{dP}$ if:

a) $Q=\frac{5P^2-9P+8}{P^2+1}$

b) $Q=\frac{6P-7}{8P-5}$

7. Differentiate, with respect to x:

a) $y=\frac{(3x+4)^3}{5x-1}$

b) $y=\left(\frac{2x+1}{3x-5}\right)^2$

8. Find the derivative of;

a) $y=2^{1-3x}$

b) $y=3\ln^2(5\sqrt{x})$

9. Find the equation of the tangent to the graph of function $f(x)$ at x_0

a) $f(x)=-2e^{3x};\ x_0=0$

b) $f(x)=\frac{x}{\ln x};\ x_0=e^2$

10. Calculate the following limits:

a) $\lim_{x\to 0}\frac{e^{2x}-\ln(x+e)}{x}$

b) $\lim_{x\to 0}\frac{e^x}{e^x+1}$

Unit 3 Applications of derivatives in Finance and in Economics

Key Unit competence: Apply differentiation in solving Mathematical problems that involve financial context such as marginal cost, revenues and profits, elasticity of demand and supply

Introductory activity

A can company produces open cans, in cylindrical shape, each with constant volume of 300 cm^3. The base of the can is made from a material that costs 50 FRW per cm^2, and the remaining part is made of material that costs 20 FRW per cm^2.

a) Express the height of one can as function of the base radius x of the can.

b) Express the total cost of the material to make a can, as function of the base radius x of the can.

c) Find the dimensions of the can that will minimize the total cost of the material to make a can.

3.1. Marginal quantities

3.1.1. Marginal cost

Learning Activity 3.1.1

A company found that the total cost y of producing x items is given by $y = 3x^2 + 7x + 12$.

a) Find the instantaneous rate of change in the total cost, when $x = 3$

b) How is the instantaneous rate of change in the total cost called?

CONTENT SUMMARY

The marginal cost is the instantaneous rate of change of the cost.

It represents the change in the total cost for each additional unit of production.

Suppose a manufacturer produces and sells *a* product. Denote C(q) to be the total cost for producing and marketing q units of the product. Thus, *C is a function of q* and it is called the *(total) cost function*. The rate of change of C with respect to q is called the *marginal cost*, that is,

$$\text{Marginal Cost} = \frac{dC}{dq}$$

Example 3.1.1.

A radio manufacturer produces x sets per week at a total cost of

$$y = \frac{1}{25}x^2 + 3x + 100 \text{ FRW.}$$

Find the marginal cost for $x = 30$

Solution:

$$\frac{d}{dx}\left(\frac{1}{25}x^2 + 3x + 100\right) = \frac{2}{25}x + 3$$

For, $x = 30$, the value of the marginal cost is $\frac{2}{25}(30) + 3 = 5.4$ per unit.

Application activity3.1.1

1. The total cost C of producing x items of a commodity is $C = 4x - x^2 + 2x^3$ FRW. Find the marginal cost of the commodity.
2. The production function of a commodity is given by $Q = 40P + 3P^2 - \frac{1}{3}P^3$, where Q is the total output and P is the number of units of input. Find the value of the marginal product for $P = 10$

3.1.2. Marginal revenue

Learning Activity 3.1.2

A firm has the following demand function: $P = 100 - Q$.

Find: a) in terms of Q, the total revenue function

b) The instantaneous rate of change of the total revenue when $Q = 11$.

CONTENT SUMMARY

If $y = C(x)$ is cost of producing x units of a product, then $R(x)$,the total revenue generated by selling x units of the product, is given by $R(x) = x.C(x)$: the product of the number of units produced by the cost of producing the **units. Then, the** marginal revenue is the instantaneous rate of change in the total revenue, that is $\frac{dR}{dx}$.

Example 3.1.2.

Given the demand function, $P = \sqrt{16 - Q}$,find the marginal total revenue for $Q = 7$

Solution:

The total revenue is $R(Q) = Q\sqrt{16 - Q}$; The marginal revenue is

$$\frac{dR}{dQ} = Q'\sqrt{16 - Q} + Q(\sqrt{16 - Q})' = \frac{-3Q + 32}{2\sqrt{16 - Q}},$$

For $Q = 7$, $\frac{dR}{dQ} = \frac{-3(7) + 32}{2\sqrt{16 - 7}} = \frac{11}{6}$

Application activity 3.1.2

1. Given the demand function $y = 30 - 2x$, find the marginal revenue
2. Find the marginal revenue associated with the supply function $P = Q^2 + 2Q + 1$ for $Q = 10$

3.2. Minimization and maximization of functions

3.2.1. Minimization of the total cost function

Learning Activity 3.2.1

Consider the following problem: A can company produces open cans, in cylindrical shape, each with constant volume of 300 cm^3. The base of the can is made from a material that costs 50 FRW per cm^2, and the remaining part is made of material that costs 20 FRW per cm^2. Assume you are the manager of the company, and you have to buy the material for constructing the can. Which question do you ask yourself regarding the dimensions of the can and the money to use for buying the material?

CONTENT SUMMARY

If function $y = f(x)$ is such that $\frac{dy}{dx} = 0$ at x_0 and $\frac{d}{dx}\left(\frac{dy}{dx}\right) > 0$ at x_0, then the function $y = f(x)$ has a local **minimum** at x_0; the minimum value of the function is $f(x_0)$.

Function $y = f(x)$ is said to be **increasing** for the values of x such that $\frac{dy}{dx} > 0$, and **decreasing** for the values of x such that $\frac{dy}{dx} < 0$,

Example 3.2.1.

Given the total cost function $P = \frac{1}{3}Q^3 - \frac{9}{2}Q^2 + 14Q + 22$, find the value of Q for which the total cost is minimum, and find the minimum total cost

Solution:

$$\frac{dP}{dQ} = Q^2 - 9Q + 14$$

$\frac{dP}{dQ} = 0$ if and only if $Q^2 - 9Q + 14 = 0$; solving this quadratic equation, we get

$Q_1 = 2$ and $Q_2 = 7$;

$\frac{d}{dQ}(Q^2 - 9Q + 14) = 2Q - 9$; its value at $Q_1 = 2$ is $2(2) - 9 = -5 < 0$;

At $Q_2 = 7$ is $2(7) - 9 = 5 > 0$;

Therefore, the minimum value of the total cost occurs for $Q = 7$, and the corresponding total cost is $\frac{1}{3}(7)^3 - \frac{9}{2}(7)^2 + 14(7) + 22 = 13.83$

Application activity 3.2.1

Find the value of Q for which the total cost is minimum, and find the minimum total cost in each of the following cases:

a) $P = \ln(Q^2 - 8Q + 20)$

b) $P = \frac{1}{3}Q^3 - \frac{17}{2}Q^2 + 60Q + 27$

3.2.2. Maximization of the total revenue function

Learning Activity 3.2.1

Consider the following problem: A company has to buy a plot for the building of its factory. The plot must have a rectangular shape with a constant perimeter of 400 meters, and the cost of the plot is constant. Assume you are the manager of the company, and you have to choose the dimensions of the rectangular plot located in a flat uniform area. Which question do you ask yourself regarding the dimensions of the plot and the area of the plot?

CONTENT SUMMARY

If the total cost function is $y = f(x)$, then the total revenue function is $R = xy$.

Suppose $\frac{dR}{dx} = 0$ at x_0 and $\frac{d}{dx}\left(\frac{dR}{dx}\right) < 0$ at x_0, then the total revenue function $R = xy$ has a local **maximum** at x_0; the maximum value of the total revenue function is $R(x_0)$.

Example 3.2.2.

Given the total cost function, $P = 5e^{-0.2Q}$, find the value of Q for which the total revenue is Given the demand function maximum, and find the maximum revenue.

Solution:

The total revenue is $R(Q) = 5Qe^{-0.2Q}$. Then, $\frac{dR}{dQ} = (5-Q)e^{-0.2Q}$;

$\frac{dR}{dQ} = 0$ if and only if $Q = 5$; $\frac{d}{dQ}[(5-Q)e^{-0.2Q}] = (-2+0.2Q)e^{-0.2Q}$;

its value at $Q = 5$ is $-e^{-1} = -\frac{1}{e} < 0$;

Therefore, the maximum value of the total revenue occurs for $Q = 5$ and the maximum value is $R(5) = \frac{25}{e} = 9.1$

Application activity 3.2.2

Given the demand function, $P = 24 - 3Q$, find the value of Q at which the total revenue is maximum, and find the maximum revenue.

3.3. Price elasticity

3.3.1. Elasticity of demand

Learning Activity 3.3.1

The price P of a commodity and the quantity demanded Q_d are related by the equation. $Q_d = f(P)$. It is observed that P increases, for a particular value of P. Determine whether Q_d will decrease or increase, or neither.

a) Determine the percentage of decrease or increase

b) Calculate $\frac{dQ_d}{dP}.\frac{P}{Q_d}$, for $P = 10$. Predict a name for $\frac{dQ_d}{dP}.\frac{P}{Q_d}$

CONTENT SUMMARY

In Economics, price elasticity ε_d measures the percentage change in quantity associated with a percentage change in price. If the quantity Q_d is related to price P by, $Q_d = f(P)$**, then the elastic**ity of demand is defined by $\varepsilon_d = \frac{dQ_d}{dP}.\frac{P}{Q_d}$.

Price elasticity of demand indicates how consumers respond to the change in the amount proposed by the producers.

If $\varepsilon_d < 0$, then Q_d and P are such that the increase in P implies the decrease in Q_d

Example 3.3.1.

The demand Q_d is related to the price P by the function, $Q_d = 650 - 5P - P^2$. Find the price elasticity of the demand at $P = 10$

Solution:

$\frac{dQ_d}{dP} = -5 - 2P$; Elasticity of the demand:

$$\varepsilon_d = \frac{dQ_d}{dP}.\frac{P}{Q_d} = [-5-2(10)]\frac{10}{650-5(10)-(10)^2} = -0.5$$

Application activity 3.3.1

Find the price elasticity of the demand if the quantity demanded Q_d and the price P are related by:

a) $Q_d = \ln\frac{100}{P^2}; P = 4$

b) $Q_d = \frac{20}{P+1}; P = 3$

3.3.2. Elasticity of supply

Learning Activity 3.3.2

A beverage company estimates that the amount Q_s of soft drink supplied per month and the quantity C bought by customers are related by a function $Q_s = f(C)$.It is observed that C increases, for a particular value of C.

Determine whether Q_s will decrease or increase, or neither;

CONTENT SUMMARY

If the quantity Q_s is related to price P **by** $Q_s = f(P)$**, then the elast**icity of demand is defined by $\varepsilon_s = \frac{dQ_s}{dP}.\frac{P}{Q_s}$. In some cases, P is given in terms of Q_s. In this case, start by making Q_s the subject of the form**ula.**

The price elasticity of supply is defined by, $\varepsilon_s = \frac{dQ_s}{dP}.\frac{P}{Q_s}$,

Where Q_s **:** quantity supplied, and P**:** amount received from consumers.

Price elasticity of supply indicates how producers respond to the change in the amount they receive from the consumers

Example 3.3.2.

The quantity supplied by the producers Q_s is related to the amount of money P received from consumers by the function $P = -2 + 5Q_s$.Find the price elasticity of supply for $P = 3$

Solution:

We have: $Q_s = \frac{1}{5}P + \frac{2}{5}$; $\frac{dQ_s}{dP} = \frac{1}{5}$; For $P = 3$, $Q_s = \frac{1}{5}(3) + \frac{2}{5} = 1$

The price elasticity of supply is $\varepsilon_s = \frac{dQ_s}{dP}.\frac{P}{Q_s} = \frac{1}{5}.\frac{3}{1} = \frac{3}{5}$,

Application activity 3.3.1

Find the price elasticity of the supply if the quantity supplied Q_s and the price P are related by:

a) $P = 3 + 4Q_s\,; P = 11$

b) $Q_s = 5Pe^{-0.2P}; P = 4$

End of unit assessment 3

1. Find the price and the quantity that will maximize the total revenue, given the demand function: $P = 12.5e^{-0.005Q}$

2. Find, the coordinates of the minimum point of function, $f(x) = \dfrac{4x}{3\ln x}$, and confirm that it is a minimum.

3. Given the total cost function, $C = Q^3 - 3Q^2 + 15Q$, find the marginal cost.

4. Given the cost function, $C = 30 - Q$, find the marginal total revenue.

5. Find the price elasticity of the demand function $Q = 150 - 15P$ at $P = 4$, where P is the price.

Unit 4 Univariate Statistics and Applications

Key Unit competence: Apply univariate statistical concepts to collect, organise, analyse, interpret data, and draw appropriate decisions

Introductory activity

1. **(a)** How do you think collecting and keeping data is important daily?

   b) In your field of study, which kind of data can a person collect? And give an example of each kind of data.

   c) Is collecting, organizing, and interpreting data helpful in making a family budget? What do you think about a national budget?

2. Suppose you have a shop selling food, and you want to know the type of food most people prefer to buy:

   i) Which statistical information will you need to collect?

   ii) How will you collect such information?

   iii) Which statistical measure will help you know the most preferred food?

3. During an accounting exam, out of 10, ten students scored the following marks: 3, 5, 6, 3, 8, 7, 8, 4, 8, 6.

   a) Determine the mean mark of the class.

   b) What is the mark that many students obtained?

   c) Compare and discuss the mean mark of the class and the mark for every student. What advice could you give to an accounting teacher?

4.1 Basic concepts in univariate statistics

4.1.1. Statistical concepts

Learning Activity 4.1.1

1. Using the internet or any other resources, do research.

   i) What do you understand by the term statistics?

   ii) What are the different branches of statistics?

   iii) What are the key terms used in statistics?

2. Suppose your company is given a market for supplying milk and fruits to all primary school students in Rwanda. If the student must choose between milk and fruits, what should a company do to ensure that it will supply what students want?

CONTENT SUMMARY

Statistics is the branch of mathematics that deals with data collection, data organization, summarization, analysis, interpretation, and drawing of conclusions from numerical facts or data.

Statistics plays a vital role in nearly all businesses and forms the backbone for all future development strategies. Every business plan starts with extensive research, which is all compiled into statistics that can influence a final decision. Statistics helps the businessman to plan production according to the taste of customers.

Branches of statistics

There are two branches of statistics, namely **descriptive**, and **inferential statistics**.

a. Descriptive statistics

Descriptive statistics deals with describing the population under study. It consists of the collection, organization, summarization, and presentation of data in a convenient and usable form.

Examples of descriptive statistics

- The average score of accounting students on the mathematics test.
- The average monthly salary of the employees in a company.
- The average age of the people who voted for the winning candidate in the last election.

b. Inferential statistics

Inferential statistics consists of generalizing from samples to populations, performing estimations and hypothesis tests, determining relationships among variables, and making predictions.

The results of the analysis of the sample can be deduced to the larger population from which the sample is taken. It consists of a body of methods for drawing conclusions or inferences about characteristics of a population based on information contained in a sample taken from the population. This is because populations are almost very large; investigating each member of the population would be impractical and expensive.

Examples of inferential statistics

- Collecting the monthly savings data of every family that constitutes your population may be challenging if you are interested in the savings pattern of an entire country. In this case, you will take a small sample of families from across the country to represent the larger population of Rwanda. You will use this sample data to calculate its mean and standard deviation.
- Suppose you want to know the percentage of people who love shopping at SIMBA supermarket. We take the sample of the population and find the proportion of individuals who love the SIMBA supermarket. With the assistance of probability, this sample proportion allows us to make a few assumptions about the population proportion.

In statistics, we generally want to study a population. Because it takes a lot of time and money to examine an entire population, we select a sample to represent the whole population.

The **population** is a collection of persons, things, or objects under the study. The population is also defined as the universe, or the entire category under consideration.

A **sample** is the portion of the population that is available, or to be made available, for analysis. A sample is also defined as a subset of the population studied. From the sample data, we can calculate a statistic.

A **statistic** is a number that represents a property of the sample. For example, if we consider one district in Rwanda to be a sample of the population of districts, then the average (mean) income generated by that one district at the end of the financial year is an example of a statistic. The statistic is an estimate of a population parameter, in this case the mean.

A **parameter** is a numerical characteristic of the whole population that can be estimated by a statistic. Since we considered all districts to be the population, then, the average (mean) income generated by district over the entire district is an example of a parameter.

Application activity 4.1.1

1. Using an example, differentiate descriptive statistics from inferential statistics.
2. We want to know the average (mean) amount of money senior five students spend at Kiziguro secondary school on school supplies that do not include books. We randomly surveyed 100 first-year students at the school. Three of those students spent 1500Frw, 2000Frw, and 2500Frw, respectively. In this example, what could be the population, sample, statistic and parameter?

4.1.2 Variables and types of variables

Learning Activity 4.1.1

Gisubizo conducted research on clients' satisfaction with bank services. She wanted to understand the relationship between clients' satisfaction and the amount of money they saved in that bank.

a) What could be the variables to consider in her research?

b) Will those variables give qualitative or quantitative information?

CONTENT SUMMARY

A **variabl**e is a characteristic of interest for each person or object in a population. A variable is a characteristic under study that takes different values for different elements. A variable, o**r random variabl**e, is a characteristic or measurement that can be determined for each member of a population. For example, if we want to know the average (mean) amount of money senior five students spend at Kiziguro secondary school on school supplies that do not include books. We randomly surveyed 100 first year students at the school. Three of those students spent 1500Frw, 2000Frw, and 2500Frw, respectively. In this example, the variable could be the amount of money spent (excluding books) by one senior five student. Let $X =$ the amount of money spent (excluding books) by one senior five student attending Kiziguro secondary school. Another example, if we collect information about income of households, then income is a variable. These households are expected to have different incomes; also, some of them may have the same income.

Note that a variable is often denoted by a capital letter like $X, \quad Y, \quad Z, \ldots$ and their values denoted by small letters for example $x, \quad y, \quad z, \ldots$ The value of a variable for an element is called an **observation** or **measurement**.

In statistics, we can collect data on a single variable or many variables. For example, if we are interested in knowing how well the company is paying its employees, we shall only collect data on the salaries of the workers in the company. In this case, we will categorize these statistics as univariate statistics. This unit only discusses univariate statistics and its application. When one variable causes change in another, we call the first variable the independent variable or explanatory variable. The affected variable is called the dependent variable or response variable. There are mainly two types of variables: qualitative variables and quantitative variables.

- **Qualitative variables**

Qualitative variables are variables that cannot be expressed using a number. They express a qualitative attribute, such as hair color, religion, race, gender, social status, method of payment, and so on. The values of a qualitative variable do not imply a meaningful numerical ordering.

Qualitative variables are sometimes referred to as **categorical variables**. For example, the variable sex has two distinct categories: 'male' and 'female.' Since the values of this variable are expressed in categories, we refer to this as a categorical variable. Similarly, the place of residence may be categorized as urban and rural and thus is a categorical variable. Categorical variables may again be described as **nominal** and **ordinal.** Ordinal variables can be logically ordered or ranked higher or lower than another but do not necessarily establish a numeric difference between each category, such as examination grades (A+, A, B+, etc., and clothing size (Extra large, large, medium, small). Nominal variables are those that can neither be ranked nor logically ordered, such as religion, sex, etc.

- **Quantitative variables**

Quantitative variables also called **numeric variables,** are those variables that are expressed in numerical terms, counted or compared on a scale. A simple example of a quantitative variable is a person's age. Age can take on different values because a person can be 20 years old, 35 years old, and so on. Likewise, family size is a quantitative variable because a family might be comprised of one, two, or three members, and so on. Each of these properties or characteristics referred to above varies or differs from one individual to another. Note that these variables are expressed in numbers, for which we call quantitative or sometimes numeric variables. A quantitative variable is one for which the resulting observations are numeric and thus possess a natural ordering or ranking.

Quantitative variables are again of two types: discrete, and continuous. Variables such as some children in a household or the number of defective items in a box are **discrete variables** since the possible scores are discrete on the scale. For example, a household could have three or five children, but not 4.52 children. Other variables, such as 'time required to complete a test' and 'waiting time in a queue in front of a bank counter,' are **continuous variables.** The time required in the above examples is a continuous variable, which could be, for example, 1.65 minutes or 1.6584795214 minutes.

Application activity 4.1.2

Suppose you have a company that sells electronic devices.

i) If you are interested in understanding how your clients are satisfied with your products. Which variable (s) will you consider in collecting the data? Is this a univariate statistics? Why?

ii) If you are interested in understanding the relationship between clients' satisfaction and educational levels. Which variable (s) will you consider in collecting the data? Is this a univariate statistics? Why?

4.1.3 Data and types of data

Learning Activity 4.1.3

Using the internet or any other resources, do research. What do you understand by the term data? Give an example.

CONTENT SUMMARY

Data are individual items of information that come from a population or sample. Data is also defined as a set of observations. Data are the values (measurements or observations) that the variables can assume. They may be numbers, or the**y may** be words. Datum is a single value. Data may come from a population or from a sample. Lower case letters like x or y generally are used to represent data values.

The observations or values that differ significantly from ot**hers are** called outliers. Outliers are at the extreme end**s of a dataset. Dataset** is a collection of data of any particular study without any manipulation. **Information** are facts about something or someone. Most data can be put into the following categories: qualitative, and quantitative.

Qualitative data are the result of categorizing or describing attributes of a population. Qualitative data are also often called categorical data. Clients' satisfaction, quality of goods, color of the car a person bought are some examples of qualitative (categorical) data. Qualitative (categorical) data are generally described by words or letters.

Quantitative data are the result of counting or measuring attributes of a population. Quantitative data are always numbers. Amount of money, number of items bought in a supermarket, and numbers of employees of the company are some examples of quantitative data. Quantitative data may be either **discrete** or **continuous.** Data is discrete if it is the result of counting (such as the number of students of a given gender in a class or the number of books on a shelf). Data is continuous if it is the result of measuring (such as distance traveled or weight of luggage). All data that are the result of counting are called **quantitative discrete data**.

These data take on only certain numerical values. If you count the number of phone calls you receive for each day of the week, you might get values such as zero, one, two, or three. Data that are not only made up of counting numbers, but that may include fractions, decimals, or irrational numbers, are called **quantitative continuous data**.

Example

You go to the supermarket and purchase three soft drinks (500ml soda, 1ml milk and 300ml juice) at 5000frw, four different kinds of fruits (apple, mango, banana and avocado) at 800frw, two different kinds of vegetables (broccoli and carrots) at 500frw, and two desserts (ice cream and biscuits) at 1000frw.

In this example,

- Number of soft drinks, different kinds of fruits, different kinds of vegetables, and desserts purchased are quantitative discrete data because you count them.
- The prices (5000frw, 800frw, 500frw, and 1000frw) are quantitative continuous data.
- Types of soft drinks, vegetable, fruits, and desserts are qualitative or categorical data.

A collection of information which is managed such that it can be updated and easily accessed is called a **database**. A software package which can be used to manipulate, validate and retrieve this database is called a **Database Management System**. For example, Airlines use this software package to book tickets and confirm reservations which are then managed to keep a track of the schedule.

Application activity 4.2.1

Describe the data and types of data used in the following study. We want to know the average amount of money spent on school uniforms annually by families with children at G.S Kayonza. We randomly survey ten families with children in the school. Ten families spent 65000Frw, 45000Frw, 65000Frw, 15000Frw, 55000Frw, 35000Frw, 25000Frw, 45000Frw, 85000Frw and 95000Frw, respectively.

4.1.4. Levels of measurement scale

Learning Activity 4.1.4

A researcher surveyed 100 people and asked them what type of place they visited (rural or urban) and how satisfied (very satisfied, satisfied, somehow satisfied, not satisfied) they were with their most recent visit to that place. Those people were also asked to provide their ages. What are the variables involved in this research?

Classify those variables according to how their data values could be categorized or measured. Is it possible to rank data values obtained from those variables?

If yes, rank them. Is it possible to find the difference between the data values of each variable?

CONTENT SUMMARY

Variables classified according to how they are categorized or measured. For example, the data could be organized into specific categories, such as major field (accounting, finance, etc.), nationality or gender. On the other hand, can the data values could be ranked, such as grade (A, B, C, D, F) or rating scale (poor, good, excellent), or they can be classified according to the values obtained from measurement, such as temperature, heights, or weights.

Therefore, we need to distinguish between them through the measurement scale used. A scale is a device or an object used to measure or quantifies any event or another object. In statistics, the variables are defined and categorized using different levels of measurements. Level of measurement or scale of measure is a classification that describes the nature of data within the values assigned to variables (Kirch, 2008).

There are four levels or scales of the measurement: Nominal, Ordinal, Interval, and Ratio.

Nominal scale

A nominal scale is used to name the categories within the variables by providing no ranking or ordering of values; it simply provides a name for each category within a variable so that you can track them among your data (Crossman, 2020). The nominal level of measurement is also known as a categorical measure and is considered qualitative in nature.

Examples

- Nominal tracking of gender (male or female)
- Nominal tracking of travel class (first class, business class and economy class).

When the classification takes ranks into consideration, the ordinal level of measurement is preferred to be used.

Ordinal scale

The ordinal level of measurement classifies data into categories that can be ordered, however precise differences between the ranks do not exist. Ordinal scales are used when people want to measure something that is not easily quantified, like feelings or opinions. Within such a scale the different values for a variable are progressively ordered, which is what makes the scale useful and informative. However, it is important to note that the precise differences between the variable categories are unknowable. Ordinal scales are commonly used to measure people's views and opinions on social issues, like quality of the products, services, or how people are satisfied with something.

Examples

- if you have a business and you wish to know how people are happy with your products or services, you could ask them a question like "How happy are you with our products or services?" and provide the following response options: "Very happy," "Somehow happy," and "Not happy."
- To test the quality of the canned product, people can use the rating scale either excellent or good or bad.

Interval scale

Unlike nominal and ordinal scales, an interval scale is a numeric one that allows for ordering of variables and provides a precise, quantifiable understanding of the differences between them.

Example

It is common to measure people's income as a range like 0Frw-100,000Frw; 100,001Frw-200,000Frw; 200,001Frw-300,000Frw, and so on. These ranges can be turned into intervals that reflect the increasing level of income, by using 1 to signal the lowest category, 2 the next, then 3, etc.

Ratio scale

The ratio scale is the interval level with additional property that there is also a natural zero starting point. In this type of scale zero means nothingness. Another difference lies in that we can attribute some of the quantities to others.

Example

The value of salary for someone is a measurement of type ratio level, where we can attribute values of wages to each other, as if to say that the person X receives a salary twice the salary of the person Y. And zero here means that the person did not receive a salary.

Application activity 4.2.1

Classify each according to the level of measurement with the interpretation of the meaning of zero if it exists.

i) Ages of the company workers (in years).

ii) Color of clothes in a shop.

iii) Temperatures inside the room (in Celsius).

iv) Nationalities of the company workers.

v) Salaries of the company employees.

vi) Weights of boxes of fruits

4.1.5. Sampling and Sampling methods

Learning Activity 4.1.5

Suppose that a certain Secondary School has 10,000 boarding students (the population).

We are interested in the average amount of money a boarding student spends on meals and accommodation in the year. Asking all 10,000 students is almost an impossible task. What would you advise that school to do so that it gets the needed information to know the average amount of money students are spending?

How will it be done so that the information the school gets represents the population?

CONTENT SUMMARY

Collecting data on entire population is costly or sometimes impossible. Therefore, a subset or subgroup of the population can be selected to represent the entire population. The process of selecting a sample from an entire population is called sampling. Since the s**ample selected is representing the wholepopulationunderstudy,thesamplemusthavet**he**samecharacteristics**as the population. There are several ways of selecting sample from the population. Some of the methods used in selecting samples are simple random sampling, stratified sampling, cluster sampling, and systematic sampling.

In stratified sampling, the population is divided into groups called strata and then takes a proportionate number from each stratum. For example, you can stratify (group) taxpayers by their Ubudehe categories then choose a proportionate simple random sample from each stratum (Ubudehe category) to get a stratified random sample. To choose a simple random sample from each category, number each member of the first category, number each member of the second category, and do the same for the remaining categories. Then use simple random sampling to choose propo**rtionate numbers** from the first category and do the same for each of the remaining categories. Those numbers picked from the first category, picked from the second category, and so on represent the members who make up the stratified sample.

In cluster sampling, the population is divided the population into clusters (groups) and then randomly select some of the clusters. All the members from these clusters are in the cluster sample.

For example, if you randomly sample your costumers by gender (males, females**, those who prefer not to say), the three groups make up the cluster sample. Number each group, and then choose four different numbers using simplerandomsampling.Allmembersofthethreegroupswiththosenumbers are the cluster sample.**

In systematic sampling, we randomly select a starting point and take every n^{th} piece of data from a listing of the population.

For example, suppose you have to do a phone survey. Your phone book contains 20,000 customers listings. You must choose 400 names for the sample. Number the population 1–20,000 and then use a simple random sample to pick a number that represents the first name in the sample. Then choose every fiftieth name thereafter until you have a total of 400 names (you might have to go back to the beginning of your phone list). Systematic sampling is frequently chosen because it is a simple method. All the above-mentioned sampling methods are random.

A type of sampling that is non-random is convenience sampling. **Convenience sampling** involves using results that are readily available.

For example, a computer software store conducts a marketing study by interviewing potential customers who happen to be in the store browsing through the available software. The results of convenience sampling may be very good in some cases and highly biased (favor certain outcomes) in others.

Application activity 4.1.5

A school account conducted a study to determine the average school fees parents pay yearly. Each parent in the following samples is asked how much fee he or she paid for each term. What is the type of sampling in each case?

a) A random number generator is used to select a parent from the alphabetical listing of all parents. Starting with that student, every 50th parent is chosen until 75 parents are included in the sample.

b) A completely random method is used to select 75 parents. Each parent has the same probability of being chosen at any stage of the sampling process.

c) The parents who have students in nursery, primary, and secondary are numbered one, two, and three, respectively. A random number generator is used to pick two of those years. All students in those two years are in the sample.

d) A sample of 100 parents having students at a school is taken by organizing the parents' names by classification as a nursery (parents whose kids are in the nursery), junior (parents whose kids are in primary), or senior (parents whose kids are in secondary), and then selecting 25 parents from each.

e) An accountant is requested to ask the first ten parents he encounters outside the school what they paid for tuition fees. Those ten parents are the sample.

4.2 Organizing and graphing data

4.2.1. Frequency table

Learning Activity 4.2.1

The weekly revenues paid (in Frw) by 20 businesspeople are below. 27000, 31000, 24000, 31000, 26000, 36000, 21000, 22000, 34000, 29000, 25000, 29000, 27000, 39000, 27000, 23000, 28000, 29000, 24000, 27000. Which revenue has been paid by many people? Represent this data in a tabular form (revenue and the number of people who paid each revenue).

CONTENT SUMMARY

Frequency tables are a great starting place for summarizing and organizing your data. Once you have a set of data, you may first want to organize it to see the frequency, or how often each value occurs in the set. Frequency tables can be used to show either quantitative or categorical data.

Example

Assume that a sample of 50 taxpayers in a district was selected to understand how taxpayers are satisfied with the taxes they are paying. The responses of those taxpayers are recorded below where (v) means very high satisfied, (s) means somewhat satisfied and (n) means not satisfied. v, n, v, n, v, s, n, n, n, n, s, s, v, n, n, n, s, n, n, s, n, n, n, s, s, s, v, v, s, v, s, v, n, n, n, n, s, v, v, v, v, v, v, v, s, v, v, v, v, v

From the recorded data above, we note that:

- Eleven of them were not satisfied with the taxes they were paying.
- Five of them were somewhat satisfied with the taxes were paying.
- Four of them were very high satisfied with the taxes were paying.

This information can be presented in a tabular form which lists the type of satisfaction (very high satisfied, somewhat satisfied, and not satisfied) and the number of students corresponding to each category. Clearly the variable is the type of satisfaction, which is qualitative variable.

Note that, each of the students belongs to one and only one of the categories. The number of students who belong to a certain category is called the frequency of that category. A frequency table shows how the frequencies are distributed over various categories.

Table 4.1: Frequency table

Type of satisfaction (variable)	Number of student (Frequency)
Very high satisfied (v)	20
Somewhat satisfied (s)	12
Not satisfied (n)	18
Sum=50	

Application activity 42.1

Consider the data on the marital status of 50 people who were interviewed.

Married	Married	Married	Married	Married Single	Divorced
Separated	Separated	Separated	Separated	Separated Single	Divorced
Single	Single	Single	Single	Single	Divorced
Single	Single	Married	Single	Separated	Divorced
Single	Separated	Married	Married	Divorced	Divorced
Divorced	Single	Single	Married	Divorced	Single
Separated	Single	Single	Single	Married	Married
Separated	Single	Single	Single	Single	Separated

Represent the above data in a frequency table.

4.2.2. Bar graph

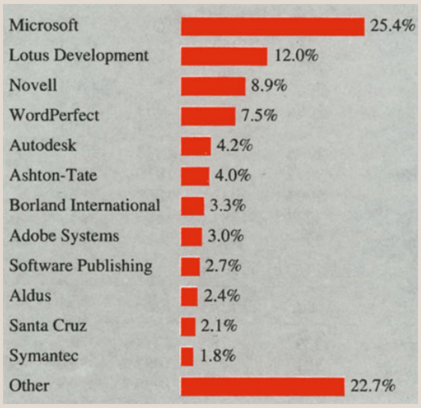
Learning Activity 4.2.1

August 27, 1991, Wall Street Journal (WSJ) article reported that the industry's biggest companies are absorbing increasing numbers of small software firms. According to WSJ, the result of this dominance by a few giants is that the industry has become tougher for software entrepreneurs to break into. The newspaper printed the chart in the accompanying figure to depict software companies' market share breakdown. From entrepreneurs to corporate giants: market share among the top 100 software companies, based on total 1990 revenue of $5.7 billion. Refer to this chart to answer the following questions:

a) List the companies in descending order of market share.

b) What is the combined market share for Lotus Development and WordPerfect?

c) What is the combined market share for Micro soft, Lotus Development, and Novell?



CONTENT SUMMARY

A bar chart or bar graph is a chart or graph that presents numerical data with rectangular bars with heights or lengths proportional to the values that they represent. The bars can be plotted vertically or horizontally. A vertical bar chart is sometimes called a line graph.

To construct a bar graph, we use the following steps:

- Represent the categories on the horizontal axis (remember to represent all categories with equal intervals).
- Mark the frequencies (or percentages) on the vertical axis.
- Draw one bar for each category that corresponds to its frequency (or percentage) on the vertical axis.

Example 1

The table below shows number of learners per class in a certain school in Rwanda.

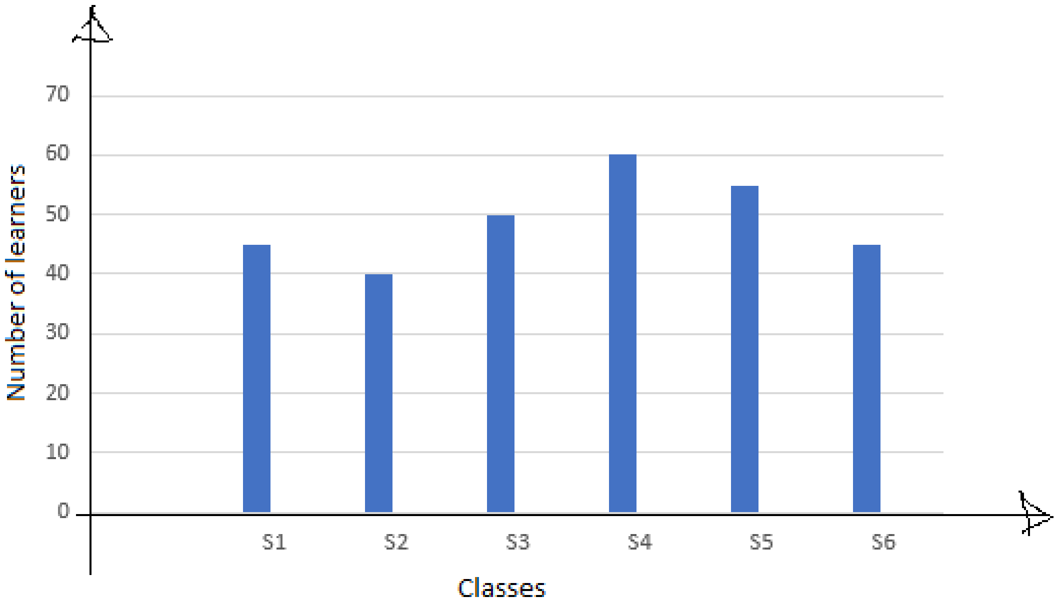
Class	S1	S2	S3	S4	S5	S6
Number of learners	45	40	50	60	55	45

a) Represent the data in a bar chart

b) How many learners are in the whole school?

Solution

a) A bar graph showing number of learners per class in a school



b) The number of learners that are in the whole school $= 45 + 40 + 50 + 60 + 55 + 45 = 295$

The school has 295 learners.

Exampe2

An insurance company determines vehicle insurance premiums based on known risk factors. If a person is considered a higher risk, their premiums will be higher. One potential factor is the color of your car. The insurance company believes that people with some color cars are more likely to get in accidents. To research this, they examine police reports for recent total loss collisions. The data is summarized in the frequency table below:

Color	Frequency
Blue	25
Green	52
Red	41
White	36
Black	39
Grey	23

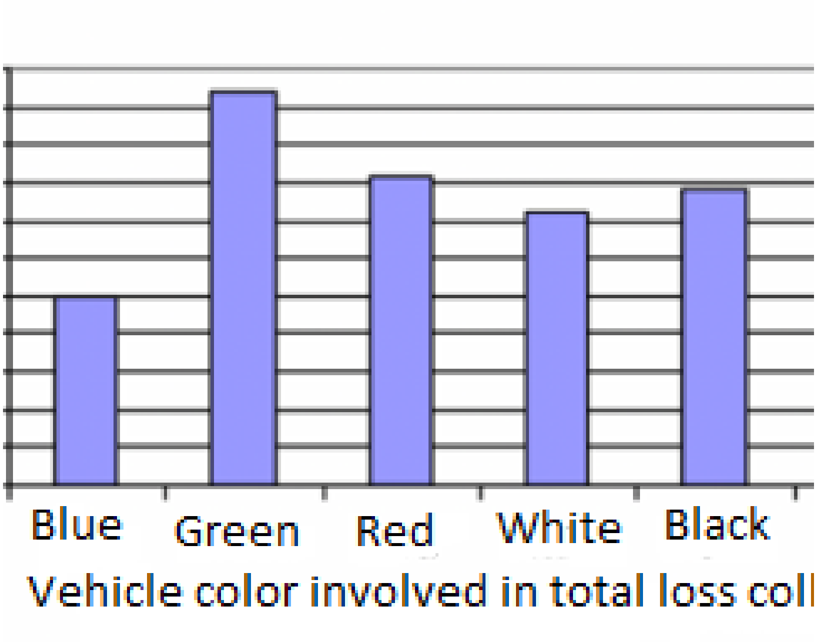
a) From the frequency table above, identify the highest frequency and the lowest frequency.

b) Present the car data on bar chart indicating frequency against vehicle color involved in total loss collision.

Solution

a) From the bar chart, the highest frequency is 52 and the lowest frequency is 23

b) The bar chart indicating frequency against vehicle color involved in total loss collision



Application activity 42.2

1. Iyamuremye is approaching retirement with a portfolio consisting of cash and money market fund investments worth 1,350,000, bonds worth 1,650,000, stocks worth 1,850,000, and real estate worth 12,000,000. Present these data in a bar chart.

2. By the end of 2022, MTN Rwanda had over 5 million users. The table below shows three age groups, the number of users in each age group, and the proportion (%) of users in each age group. Construct a bar graph of this data.

Age groups	Number of MTN users	Proportion (%) of MTN users
13–25	2,250,000	45%
26–44	1,800,000	36%
45–64	950,000	19%

4.2.3 Histogram and polygon

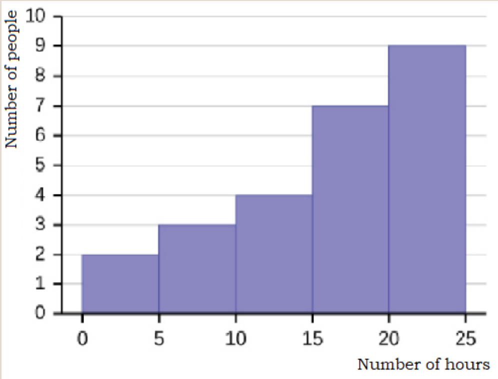
Learning Activity 4.2.3

The graph below indicates the number of hours people work during a week. The vertical axis represents the number of people, while the horizontal axis represents the number of hours people spend at work.

a) How many people spend more hours at work? How many hours do those people spend?

b) In total, how many people participated in this study?

c) What is the name of this graph?



CONTENT SUMMARY

After you have organized the data into a frequency distribution, you can present them in graphical form. The purpose of graphs in statistics is to convey the data to the viewers in pictorial form. It is easier for most people to comprehend the meaning of data presented graphically than data presented numerically in tables or frequency distributions.

The three most commonly used graphs in research are

a) The histogram.

b) The frequency polygon.

c) The cumulative frequency graph or ogive (pronounced o-jive).

a. The Histogram

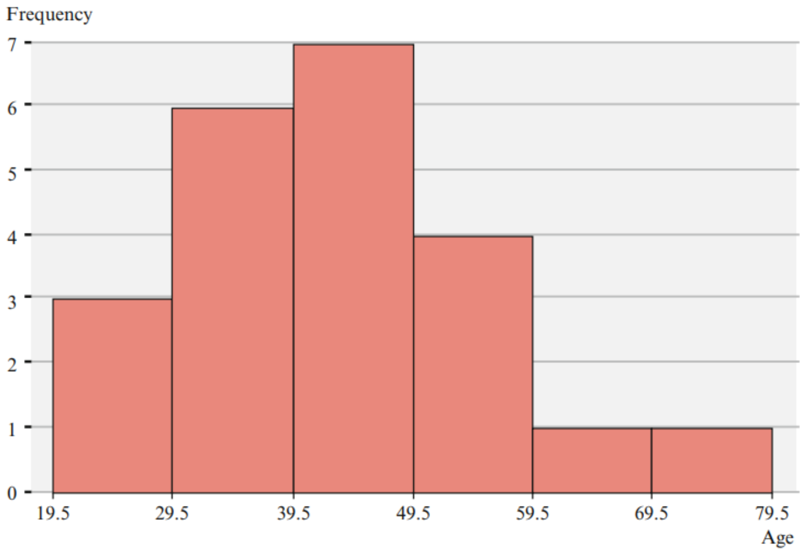
The **histogram** is a graph that displays the data by using contiguous vertical bars (unless the frequency of a class is 0) of various heights to represent the frequencies of the classes.

Example1: suppose the age distribution of personnel at a small business is: 25, 24, 29, 20, 32, 39, 36, 30, 30, 39, 40, 42, 45, 47, 48, 43, 49, 50, 54, 58, 50, 65, 79.

Form classes by grouping ages of these personnel in categories as follows: 20-29, 30-35, 39, 40-49, 50-59, 60-69, 70-79.

For each group, write the number of times numbers in that group are occurring. To construct a histogram, we need to enter a scale on the horizontal axis. Because the data are discrete, there is a gap between the class intervals, say between 20 and 29 and 30–39.

In such a case, we will use the midpoint between the end of one class and the beginning of the next as our dividing point. Between the 20–29 interval and 30–39 interval, the dividing point will be $\frac{(20+29)}{2}=19.5$, $\frac{(29+30)}{2}=29.5$ respectively. We find the dividing point between the remaining classes similarly.



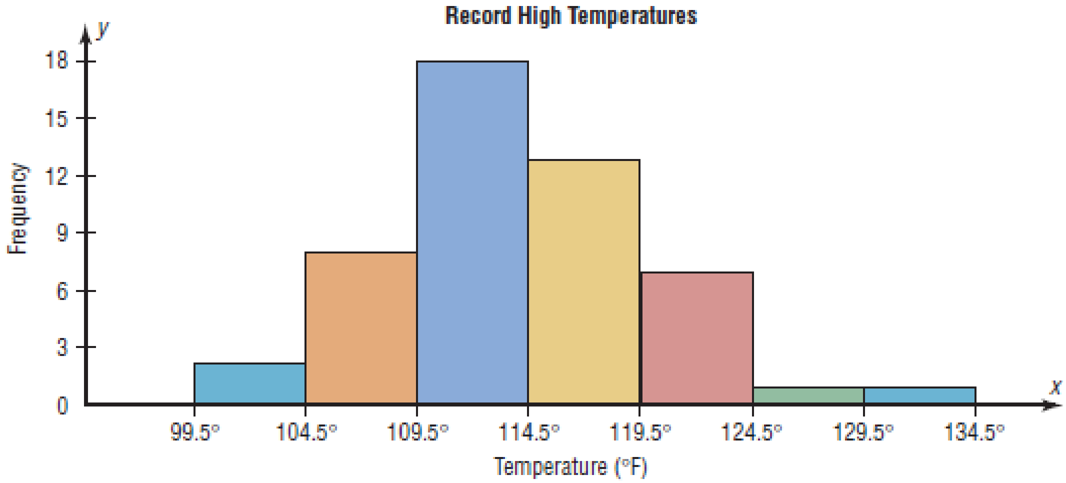
Example2: Construct a histogram to represent the data shown for the record high temperatures for each of the 50 states.

Class boundaries	Frequency
99.5-104.5	2
104.5-109.5	8
109.5-114.5	18
114.5-119.5	13
119.5-124.5	7
124.5-129.5	1
129.5-134.5	1

Step 1: Draw and label the x and y axes. The x axis is always the horizontal axis, and the y axis is always the vertical axis.

Step 2: Represent the frequency on the y axis and the class boundaries on the x axis.

Step 3: Using the frequencies as the heights, draw vertical bars for each class. See Figure below



b. The Frequency Polygon

The **frequency polygon** is a graph that displays the data by using lines that connect points plotted for the frequencies at the midpoints of the classes. The frequencies are represented by the heights of the points.

Example:

Using the frequency distribution given in **Example 2**, construct a frequency polygon

Step 1: Find the midpoints of each class. Recall that midpoints are found by adding the upper and lower boundaries and dividing by 2:

$$\frac{99.5+104.5}{2}=102, \quad \frac{104.5+109.5}{2}=107 \quad \text{and so on.}$$

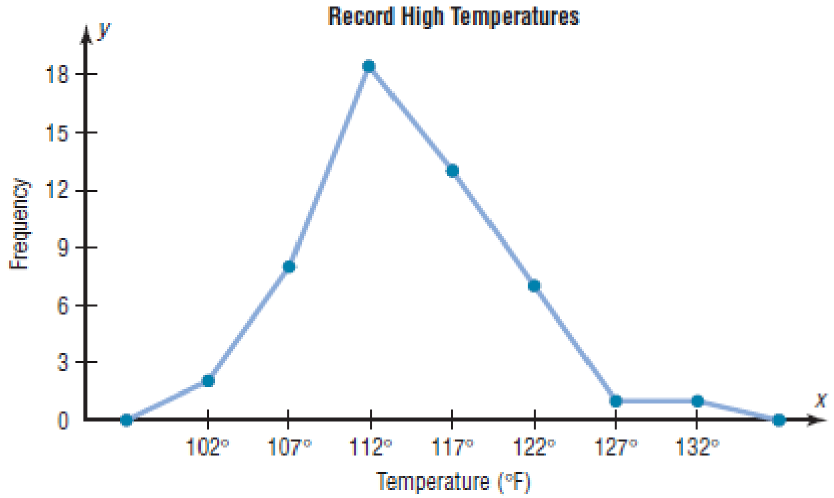
The midpoints are:

Class boundaries	Midpoints	Frequency
99.5–104.5	102	2
104.5–109.5	107	8
109.5–114.5	112	18
114.5–119.5	117	13
119.5–124.5	122	7
124.5–129.5	127	1
129.5–134.5	132	1

Step 2: Draw the x and y axes. Label the x axis with the midpoint of each class, and then use a suitable scale on the y axis for the frequencies.

Step 3: Using the midpoints for the x values and the frequencies as the y values, plot the points.

Step 4: Connect adjacent points with line segments. Draw a line back to the x axis at the beginning and end of the graph, at the same distance that the previous and next midpoints would be located, as shown in figure.



The frequency polygon and the histogram are two different ways to represent the same data set. The choice of which one to use is left to the discretion of the researcher.

c. The cumulative frequency graph or Ogive.

The **ogive** is a graph that represents the cumulative frequencies for the classes in a frequency distribution.

Step 1: Find the cumulative frequency for each class.

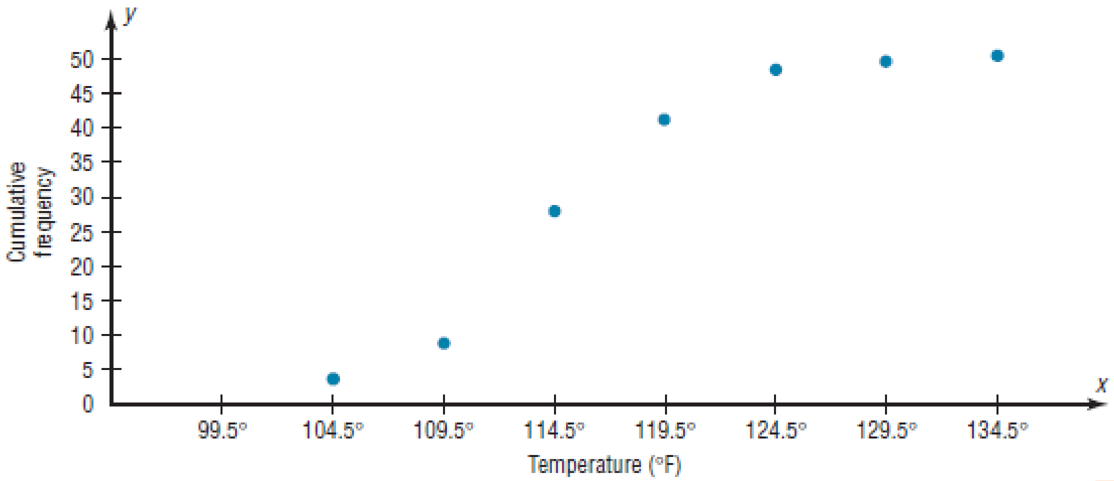
	Cumulative frequency
Less than 99.5	0
Less than 104.5	2
Less than 109.5	10
Less than 114.5	28
Less than 119.5	41
Less than 124.5	48
Less than 129.5	49
Less than 134.5	50

Step 2: Draw the x and y axes. Label the x axis with the class boundaries. Use an appropriate scale for the y axis to represent the cumulative frequencies.

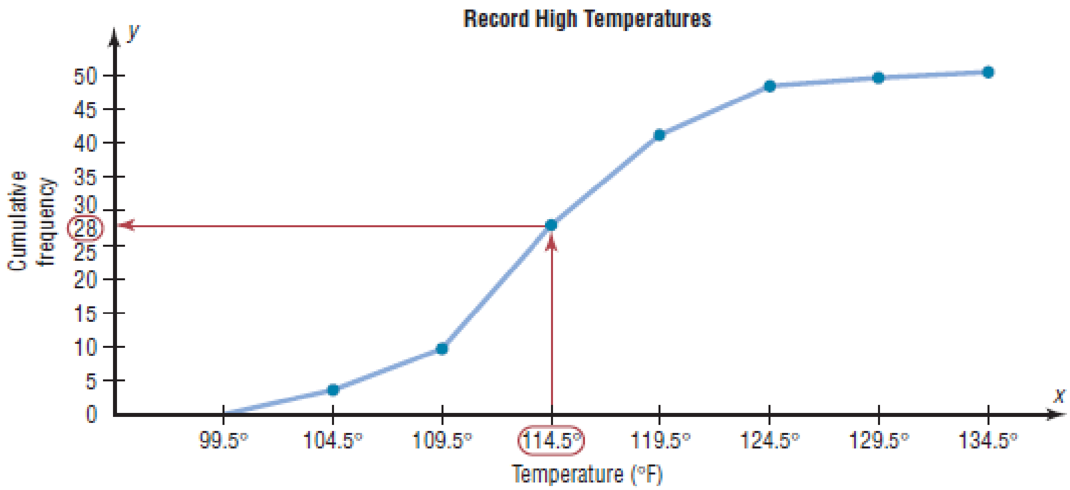
(Depending on the numbers in the cumulative frequency columns, scales such as 0, 1, 2, 3, . . . , or 5, 10, 15, 20, . . . , or 1000, 2000, 3000, . . . can be used.

Do *not* label the *y* axis with the numbers in the cumulative frequency column.) In this example, a scale of 0, 5, 10, 15, . . . will be used.

Step 3: Plot the cumulative frequency at each upper class boundary, as shown in Figure below. Upper boundaries are used since the cumulative frequencies represent the number of data values accumulated up to the upper boundary of each class.



Step 4: Starting with the first upper class boundary, 104.5, connect adjacent points with line segments, as shown in the figure. Then extend the graph to the first lower class boundary, 99.5, on the *x* axis.



Cumulative frequency graphs are used to visually represent how many values are below a certain upper class boundary. For example, to find out how many record high temperatures are less than 114.5 ^{0}F, locate 114.5 ^{0}F on the x axis, draw a vertical line up until it intersects the graph, and then draw a horizontal line at that point to the y axis. The y axis value is 28, as shown in the figure.

Application activity 42.3

Consider the following data:

45,50,55,60,65,70,75,47,51,56,61,66,71,76,48,52,57,62,67,72,77,49,53,58,63,68,73,78,49,54,59,64,68,74,49,51,55,61,68,71,51,56,61,69,71,52,56,62,66,72,53,57,62,67,72,54,58, 63,67,74,58,63,68,58,64,68,59,64,69,55,64,69,56,64,68,61, 61,62,62,63.

Then:

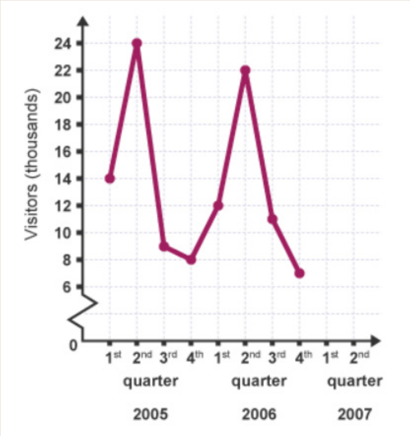
a) Make this data in a frequency distribution table with Class boundaries width equal to 5 and containing Class boundaries, Midpoints, Frequencies, Relative frequencies, Percentages, and Cumulative frequencies.

b) Draw the histogram for the frequencies, relative frequencies, percentages, and percentage frequencies in the distribution table.

4.2.4 Time series graph

Learning Activity 4.2.4

The graph below shows the number of buyers per quarter (per three months) who have visited a supermarket.



From the graph,

a) In which quarter did few people visit the supermarket?

b) How many people did buy at the supermarket in the second quarter of 2005?

c) In total, how many buyers did visit the supermarket from 2005 to 2007?

CONTENT SUMMARY

In most graphs and charts, the independent variable is plotted on the horizontal axis (the $X-axis$) and the dependent variable on the vertical axis (the $Y-axis$).

A **time series** is defined as having the independent variable of time and the dependent variable as the value of the variable being studied.

A time series graph is a **line graph that shows data such as measurements, sales or frequencies over a given time**.

Frequently, "time" is plotted along the x-axis. Such a graph is known as a time-series graph because on it, changes in a dependent variable (such as GDP: Gross Domestic Production, inflation rate, or stock prices) can be traced over time.

They can be used to show a pattern or **trend in the data** and are useful for making **predictions** about the future such as weather forecasting or financial growth.

To create the time series graph,

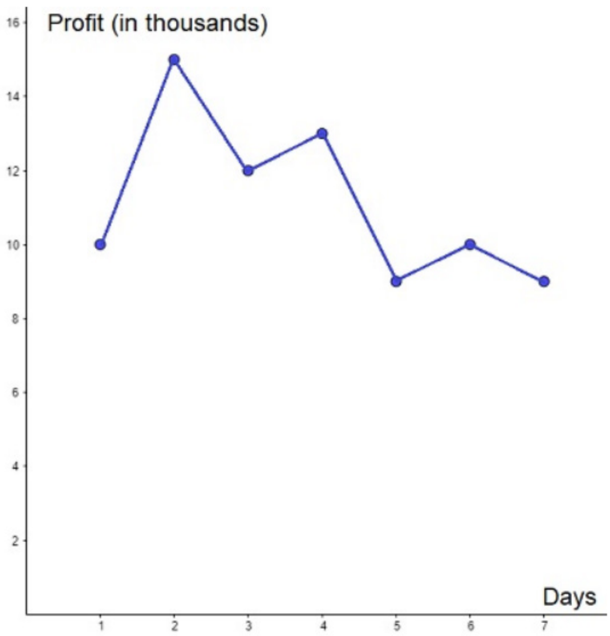
- Start off by labeling the time-axis in chronological order.
- Label the vertical axis and horizontal axis. The **horizontal axis** always shows the **time**, and the **vertical axis** represents the **variable being recorded against time**.
- After labelling, plot the points given in the data set.
- Finish the graph by connecting the dots with straight lines.

Example

In a week, a certain company is making a profit of 10000 FRW on the first day, 15000FRW on the second day, 12000FRW on the third day, 13000FRW on the fourth day, 9000FRW on the fifth day, 10000FRW on the sixth day, and 9000FRW on the seventh day. In a tabular form, this can be presented as

Day	1	2	3	4	5	6	7
Profit (FRW)	10000	15000	12000	13000	9000	10000	9000

The time series graph is



Application activity 4.2.4

The following data shows the Annual Consumer Price Index each month for ten years. Construct a time series graph for the Annual Consumer Price Index data only.

Year	Aug	Sep	Oct	Nov	Dec	Annual
2003	184.6	185.2	185.0	184.5	184.3	184.0
2004	189.5	189.9	190.9	191.0	190.3	188.9
2005	196.4	198.8	199.2	197.6	196.8	195.3
2006	203.9	202.9	201.8	201.5	201.8	201.6
2007	207.917	208.490	208.936	210.177	210.036	207.342
2008	219.086	218.783	216.573	212.425	210.228	215.303
2009	215.834	215.969	216.177	216.330	215.949	214.537
2010	218.312	218.439	218.711	218.803	219.179	218.056
2011	226.545	226.889	226.421	226.230	225.672	224.939
2012	230.379	231.407	231.317	230.221	229.601	229.594

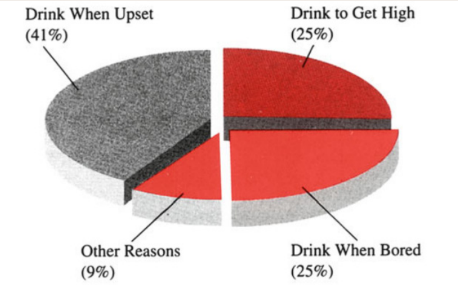
4.2.5 Pie chart

Learning Activity 4.2.5

The chart below presents data on why teenagers drink. Use the information shown in the chart to answer the following questions:

a) For what reason do the highest numbers of teenagers drink?

b) What percentage of teenagers drink because they are bored or upset?



CONTENT SUMMARY

A pie chart, sometimes called a circle chart, is a way of summarizing a set of data in circular graph. This type of chart is a circle that is divided into sections or wedges according to the percentage of frequencies in each category of the distribution. Each part is represented in degrees.

To present data using pie chart, the following steps **are r**espected:

Step 1: Write all the data into a table and add up all the values **to get** a total.

Step 2: To find the values in the form of a percentage divide each value by the total and multiply by 100. That means that each frequency must also be converted to a percentage by using **the formula**

$$\% = \frac{f}{n}.100\%$$

Step 3: To find how many degrees for each pie sector we need, we take a full circle of 360° and use the formula:

$$Angle\ for\ sector\ S = \frac{Frequency\ of\ S \times 360^0}{Total\ frequency}$$

Since there are 360^0 in a circle, the frequency for each class must be converted into a proportional part of the circle. This conversion is done by using the formula:

$$Degrees = \frac{f}{n}.360^0$$

where f is frequency for each class and n is the sum of the frequencies. Hence,

the following conversions are obtained. The degrees should sum to 360.

Step 4: Once all the degrees for creating a pie chart are calculated, draw a circle (pie chart) using the calculated measurements with the help of a protractor, and label each section with the name and percentages or degrees.

Example.

1. In the summer, a survey was conducted among 400 people about their favourite beverages: 2% like cold-drinks, 6% like Iced-tea, 12% like Cold-coffee, 24% like Coffee and 56% like Tea.

   a) How many people like tea?

   b) How many more people like coffee than cold coffee?

c) What is the total central angle for iced tea and cold-drinks?

d) Draw a pie chart to represent the provided information.

Solution:

a) Total number of people = 400

Number of people like tea $= 400 \times \frac{56}{100} = 224$

a) Number of people like coffee $= 400 \times \frac{24}{100} = 96$

Number of people like cold-coffee $= 400 \times 12 = 48$

Number of people like coffee more than cold-coffee $= 96 - 48 = 48$

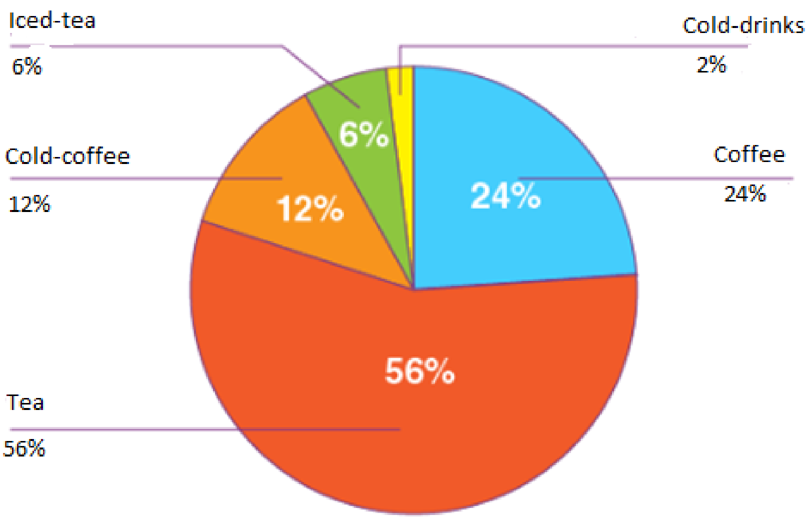
a) Number of people like iced-tea $= 400 \times \frac{6}{100} = 24$

Number of people like cold-drinks $= 400 \times \frac{2}{100} = 8$

Central angle for iced-tea $= \frac{24}{400} \times 360^0 = 21.6^0$

Central angle for cold-drinks $= \frac{8}{400} \times 360^0 = 7.2^0$ = 8/400 × 360° = 7.2°

Total central angle $= 21.6^0 + 7.2^0 = 28.8^0$ = 21.6° + 7.2° = 28.8°.



Example 2:

A person spends his time on different activities daily (in hours):

Activity	Office work	exercise	Travelling	Watching shows	Sleeping	Miscellaneous
Number of hours	9	1	2	3	7	2

a) Find the central angle and percentage for each activity.
b) Draw a pie chart for this information
c) Use the pie chart to comment on these findings.

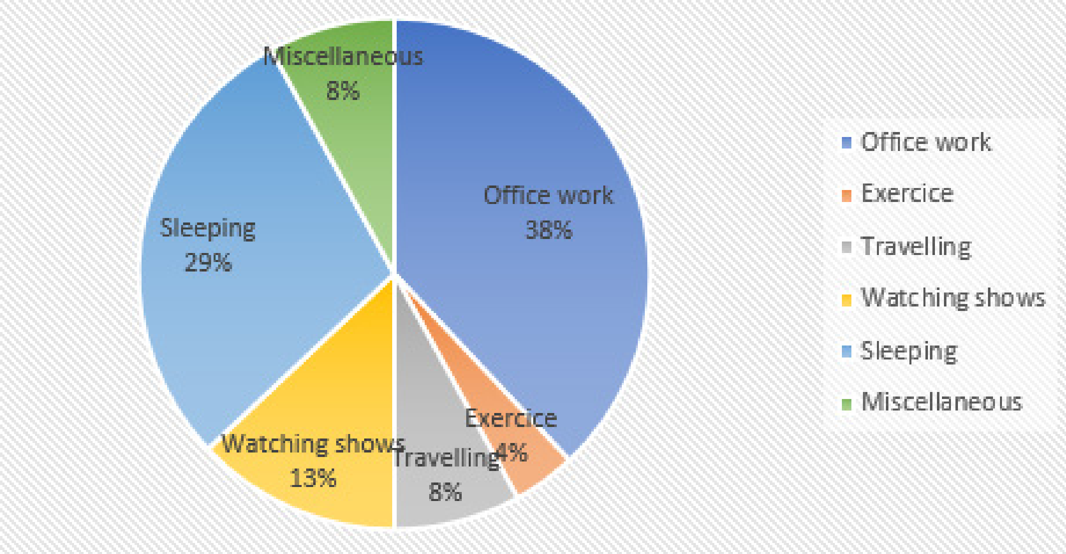
Solution

a) Central angles are calculated by using the formula:

$Degrees = \frac{f}{n} \times 360^0$ **and percentages calculated by using the formula:** $Percentage = \frac{f}{n} \times 100\%$

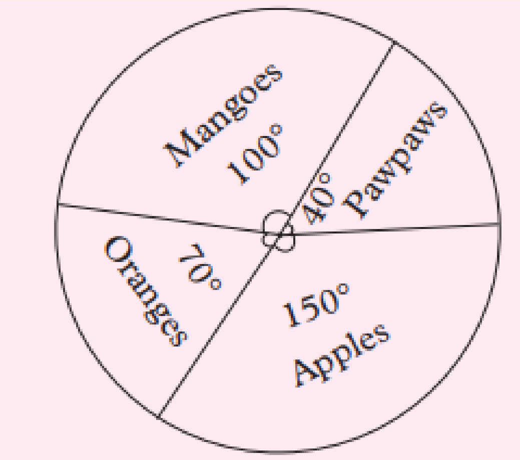
Activity	Number of hours	Central angle	Percentage
Office work	9	$\frac{9}{24} \times 360^0 = 135^0$	$\frac{9}{24} \times 100\% = 37.5\%$; 38%
Exercise	1	$\frac{1}{24} \times 360^0 = 15^0$	$\frac{1}{24} \times 100\% = 4.16\%$; 4%
Travelling	2	$\frac{2}{24} \times 360^0 = 30^0$	$\frac{2}{24} \times 100\% = 8.33\%$; 8%
Watching shows	3	$\frac{3}{24} \times 360^0 = 45^0$	$\frac{3}{24} \times 100\% = 12.5\%$; 13%
Sleeping	7	$\frac{7}{24} \times 360^0 = 105^0$	$\frac{7}{24} \times 100\% = 29.16\%$; 29%
Miscellaneous	2	$\frac{2}{24} \times 360^0 = 30^0$	$\frac{2}{24} \times 100\% = 8.33\%$; 8%
Total	24	360^0	100

b) Using a protractor, graph each section and write its name and corresponding percentage, as shown in the Figure below



Application activity 4.2.5

After selling fruits in a market, Aisha had a total of 144 fruits remaining. The pie chart below shows each type of fruit that remained.



a) Find the total cost of mangoes and pawpaws if a mango sells at 30 FRW and pawpaw at 160 FRW each.

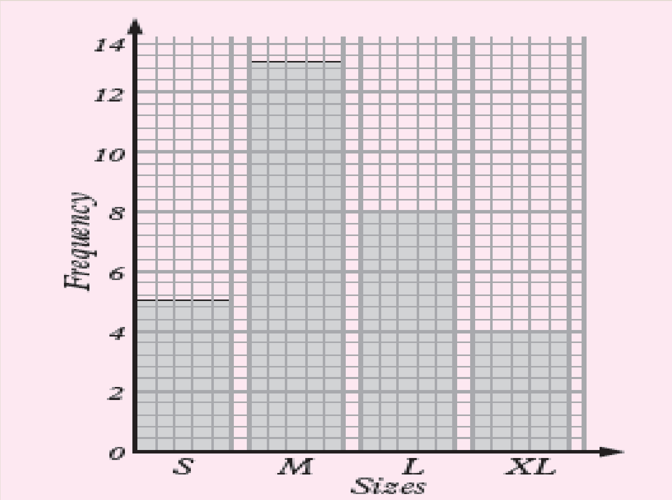
b) Which types of fruit remained the most?

c) Draw a frequency table to display the information on the pie chart.

4.2.6 Graph interpretation

Learning Activity 4.2.4

The graph below shows the sizes of sweaters worn by 30 year 1 students in a certain school. Observe it and interpret it by answering the questions below:



a) How many students are with small size?

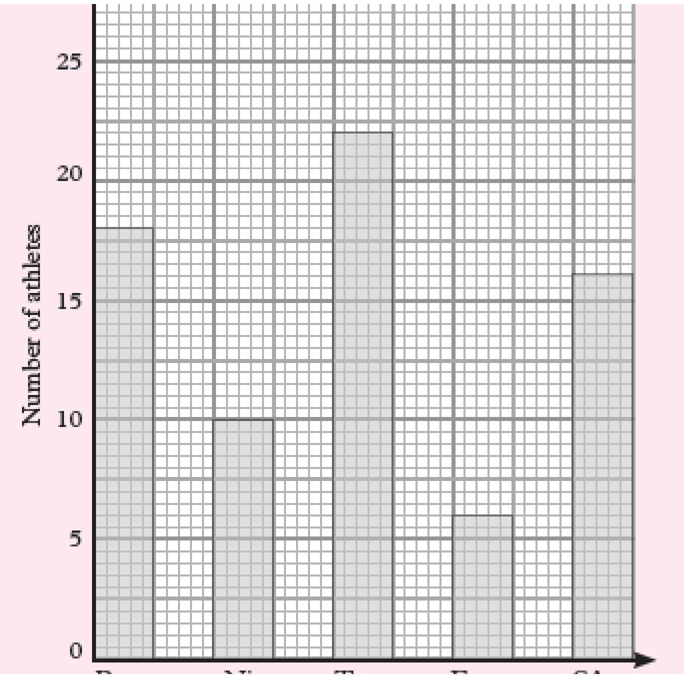
b) How many students with medium size, large size and extra large size are there?

CONTENT SUMMARY

Once data has been collected, they may be presented or displayed in various ways including graphs. Such displays make it easier to interpret and compare the data.

Examples

1. The bar graph shows the number of athletes who represented five African countries in an international championship.



a) What was the total number of athletes representing the five countries?

b) What was the smallest number of athletes representing one country?

c) What was the most number of athletes representing a country?

d) Represent the information on the graph on a frequency table.

Solution:

We read the data on the graph:

a) Total number of athletes are: 18 + 10 + 22 + 6 + 16 = 72 athletes

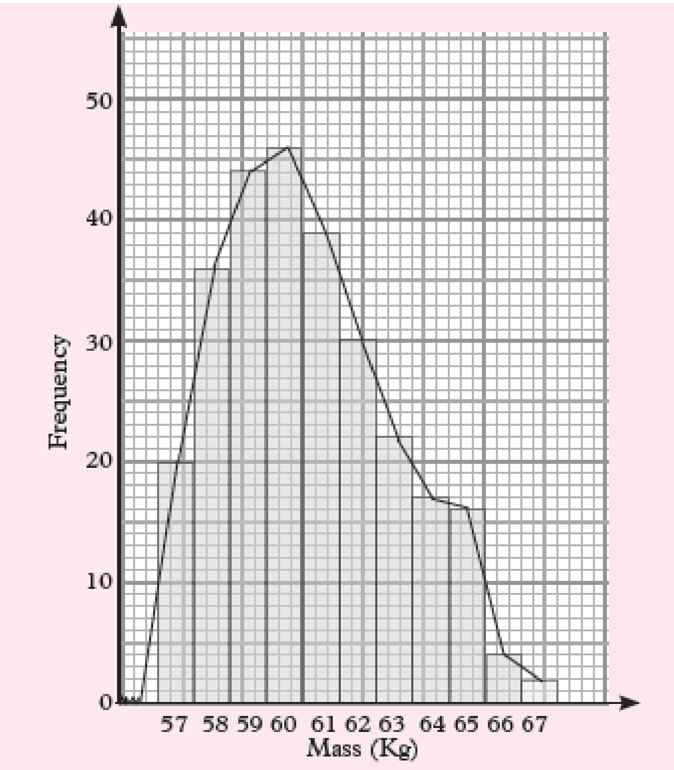
b) 6 athletes

c) 22 athletes

d) Representation of the given information on the graph on a frequency table.

Country	Number of athletes
Rwanda	18
Nigeria	10
Tanzania	22
Egypt	6
South Africa	16
Total	**72**

2. Use a scale vertical scale 2cm: 10 students and Horizontal scale 2cm: 10 represented on histogram below to answers the questions that follows



a) estimate the mode
b) Calculate the range

Solution:

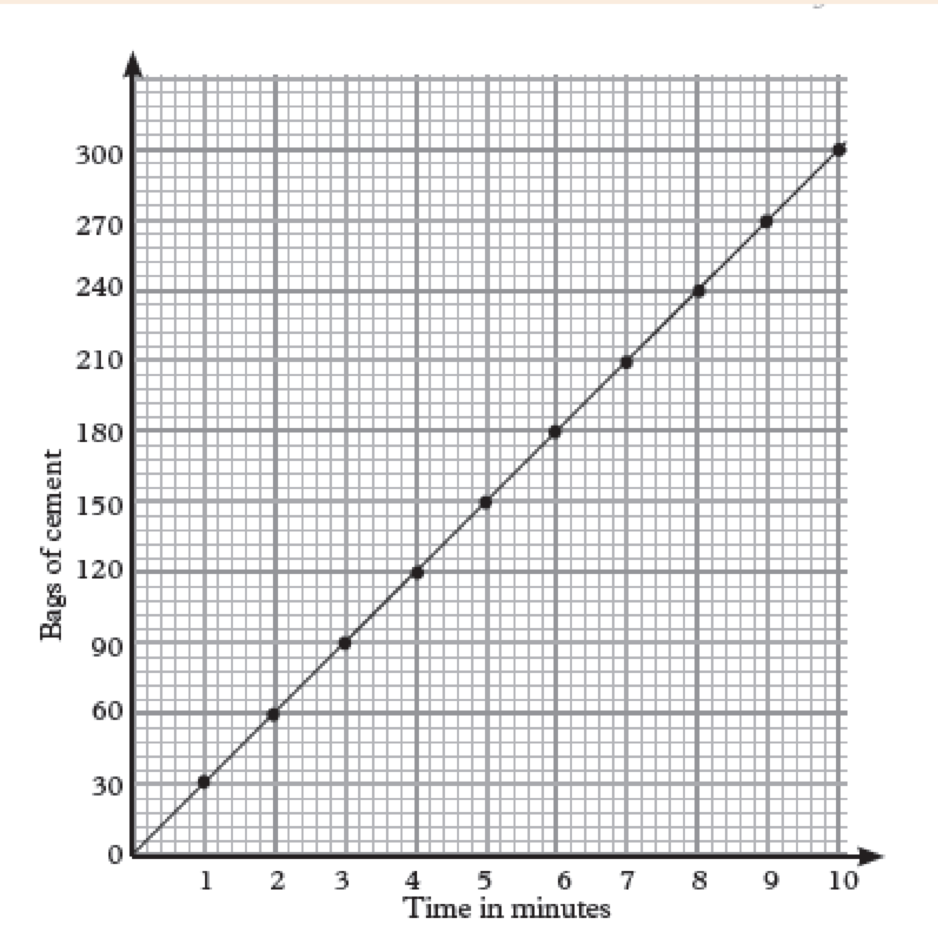
a) To estimate the mode graphically, we identify the bar that represents the highest frequency. The mass with the highest frequency is 60 kg. It represents the mode.

b) The highest mass = 67 kg and the lowest mass = 57 kg

Then, The range=highest mass-lowest mass= $67kg - 57kg = 10kg$

Application activity 4.2.5

The line graph below shows bags of cement produced by CIMERWA industry cement factory in a minute.



a) Find how many bags of cement will be produced in: 8 minutes, 3 minutes12 seconds, 5 minutes and 7 minutes.

b) Calculate how long it will take to produce: 78 bags of cement.

c) Draw a frequency table to show the number of bags produced and the time taken.

4.3 Numerical descriptive measures

4.3.1 Describing data using mean, median, and mode

Learning Activity 4.3.1

Consider a portfolio that has achieved the following returns: Q_1=+10%, Q_2=-3%, Q_3=+8%, Q_4=+12%, Q_5=-7%, Q_6=+12% and Q_7=+3% over seven quarters.

a) What is the average return on investment?

b) Which return of the portfolio is in the middle?

c) Which return of the portfolio that has been achieved frequently?

CONTENT SUMMARY

A measure of central tendency is very important tool that refer to the centre of a histogram or a frequency distribution curve. There are three measures of central tendency:

- Mean
- Median
- Mode

Difference between the mean, median and mode

- Mean is the average of a data set.
- The median is the middle value in a set of ranked observations. It is also defined as the middle value in a list of values arranged in either ascending or descending orders.
- Mode is the most frequently occurring value in a set of values.

How to find mean

The most used measure of central tendency is called mean (or the average). Here the main of interest is to learn how to calculate the mean when the data set is raw data.

The following steps are used to calculate the mean:

Step 1: Add the numbers

Step 2: Count how many numbers there are in the data set

Step 3: Find the mean by dividing the sum of the data values by the number of data values.

Mathematically, mean is calculated as follows:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$, where n is the number of observations in the dataset,

x_i are observations.

Or $\bar{x} = \frac{1}{n}\sum xf_i$

Here, the mean can also be calculated by multiplying each distinct value by its frequency and then dividing the sum by the total number of data values.

How to find median

- Rank the given data sets (in increasing or decreasing order)
- Find the middle term for the ranked data set that obtained in step 1.
- The value of this term represents the median.

In general form, calculating the median depends on the number of observations (even or odd) in the data set, therefore applying the above steps requires a general formula.

Consider the ranked data $x_1, x_2, x_3, \ldots, x_n$ the formula for calculating the median for the two cases (even and odd) is given by:

If n is odd, Median $= x_{\left(\frac{n+1}{2}\right)}$, or

median is given by $\left(\frac{n+1}{2}\right)^{th}$ number which is located on this position

If n is even, Median $= \frac{x_{\left(\frac{n}{2}\right)} + x_{\left(\frac{n}{2}+1\right)}}{2}$, or

Median is given by $\frac{1}{2}\left[\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}\right]$, then the median is a half of the sum of number located on those two positions.

Example 1

To understand the three statistical concepts, consider the following example: A Supermarket recently launched a new mint chocolate chip ice cream flavour. They want to compare customer traffic numbers to their store in the past seven days since the launch to understand whether their new offering intrigued customers. Here is the customer data from last week: Monday =92 customers, Tuesday =92 customers, Wednesday =121 customers, Thursday =120 customers, Friday = 132 customers, Saturday = 118 customers, and Sunday =128 customers. To make sense of this data, we can calculate the average:

- Find the sum by adding the customer data together, 92 + 92 + 118 + 120 + 121 + 128 + 132 = 803
- Number of days is equal to 7.
- Mean or average is $\frac{803}{7} = 114.714$. This means that mean average of customers in the past week is 115 customers.

The mode is 92 customers because on Monday and Tuesday, 92 customers were received. To find the median, we need to arrange data as follows: 92, 92, 118, 120, 121, 128, 132. Then, the middle value is 120. Therefore, the median is 120.

By using the formula, n is equal to 7 which is odd. The median $= x_{\left(\frac{n+1}{2}\right)}$ is $x_{\left(\frac{7+1}{2}\right)} = x_4$.

The value at the fourth position in the ranked data above is 120. Hence, median is 120.

Example 2

Calculate the mean of the pocket money of some 5 students who get

$2500\ FRW,\ 4000\ FRW,\ 5500\ FRW,\ 7500\ FRW\ and\ 3000\ FRW.$

Sum all the pocket money of five students

$= (2500 + 4000 + 5500 + 7500 + 3000)\ FRW = 22500 FRW.$

Divide the sum by the number of students $= 22500/5 = 4500 FRW$.

The mean of the pocket money of 5 students is $4500 FRW$.

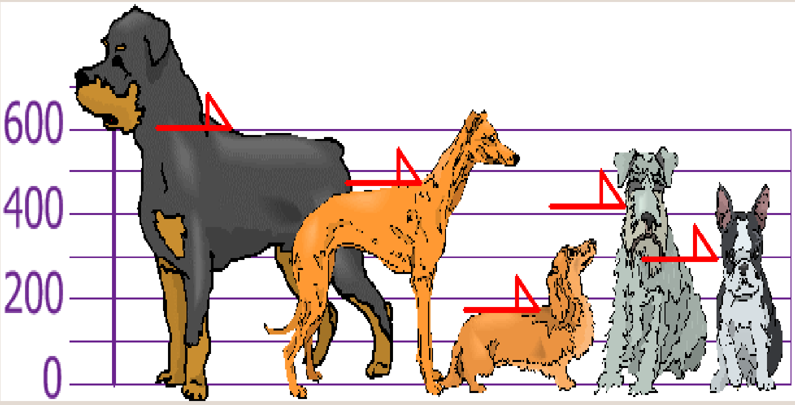
Application activity 4.3.1

Find the average and median monthly salary (FRW) of all six secretaries each month earn (in thousands) 104, 340, 140, 185, 270, and 258 each, respectively.

4.3.2 Summarizing data using variance, standard deviation, and coefficient of variation

Learning Activity 4.3.2

You and your friends have just measured the heights of your dogs (in millimeters):



The heights (at the shoulders) are 600mm, 470mm, 170mm, 430mm, and 300mm.

a) Work out the mean height of your dogs.

b) For each height subtract the mean height and square the difference obtained (the squared difference).

c) Work out the average of those squared differences. What do you notice about the average?

CONTENT SUMMARY

The following are measures of variation:

- Variance
- Standard deviation
- Range
- Mean deviation.

Variance

Variance measures how far a set of numbers is spread out. A variance of zero indicates that all the values are identical. Variance is always non-negative: a small variance indicates that the data points tend to be very close to the mean and hence to each other, while a high variance indicates that the data points are very spread out around the mean and from each other.

- For the population, the variance is denoted and defined by:

$\sigma^2 = \frac{\Sigma(x-\mu)^2}{N}$, where x is individual value, μ is the population mean, and N is the population size.

- For the sample, the variance is denoted and defined by:

$S^2 = \frac{\Sigma\left(x-\bar{x}\right)^2}{n-1}$, where x is individual value, $\bar{x}$ is the sample mean, and n is the sample size.

How to find variance

To calculate the variance follow these steps:

- Work out the mean (the simple average of the numbers)
- Then for each number: subtract the Mean and square the result (the *squared difference*).
- Then work out the average of those squared differences.

Example1:

The heights (in meters) of six children are 1.42, 1.35, 1.37, 1.50, 1.38 and 1.30.

Calculate the mean height and the variance of the heights.

Solution:

$$Mean = \frac{1}{6}(1.42+1.35+1.37+1.50+1.38+1.30) = 1.39\,m$$

$$Variance = \frac{1}{6}\left(1.42^2+1.35^2+1.37^2+1.50^2+1.38^2+1.30^2\right)-(1.39)^2 = 0.00386\,m$$

Example2:

The number of customers served lunch in a restaurant over a period of 60 days is as follows:

Number of customers served lunch	20 – 29	30 – 39	40 – 49	50 – 59	60 – 69	70 – 79
Number of days in the 60-day period	6	12	16	14	8	4

Find the mean and variance of the number of customers served lunch using this grouped data.

Solution

To find the mean from grouped data, first we determine the mid-interval values for all intervals;

Intervals	Mid-interval values (xi)	Frequency (fi)	$f_i x_i$	$f_i x_i^2$
20-29	24.5	6	147	3601.5
30-39	34.5	12	414	14283.0
40-49	44.5	16	712	31684.0
50-59	54.5	14	763	41583.5
60-69	64.5	8	516	33282.0
70-79	74.5	4	298	22201.0
		$\sum = 60$	$\sum = 2850$	$\sum = 146635$

The mean is $\bar{x} = \frac{2850}{60} = 47.5$

$$Variance = \frac{146635}{60} - (47.5)^2 = 187.67\ .$$

Standard deviation

A most used measure of variation is called standard deviation denoted by (σ for the population and S for the sample). The numerical value of this measure helps us how the values of the dataset corresponding to such measure are relatively closely around the mean.

Lower value of the standard deviation for a data set, means that the values are spread over a relatively smaller range around the mean. Larger value of the standard deviation for a data set means that the values are spread over a relatively smaller range around the mean.

How to find standard deviation

Take a square root of the variance. The population standard deviation is defined as: square root of the average of the squared differences from the population mean.

$\sigma = \sqrt{\sigma^2} = \sqrt{\dfrac{\Sigma(x-\mu)^2}{N}}$, where x is individual value, μ is the population mean, and N is the population size.

The sample standard deviation is defined as: square root of the average of the squared differences from the sample mean.

$S = \sqrt{S^2} = \sqrt{\dfrac{\Sigma\left(x-\bar{x}\right)^2}{n-1}}$, where x is individual value, $\bar{x}$ is the sample mean, and n is the sample size.

Example:

The six runners in a 200 meter race clocked times (in seconds) of 24.2, 23.7, 25.0, 23.7, 24.0, 24.6.

Find the mean and standard deviation of these times.

Solution

$$\bar{x} = \frac{24.2+23.7+25.0+23.7+24.0+24.6}{6} = 24.2 \text{ seconds}$$

$$\sigma=\sqrt{\frac{(24.2-24.2)^2+(23.7-24.2)^2+(25.0-24.2)^2+(23.7-24.2)^2+(24.0-24.2)^2+(24.6-24.2)^2}{6}}$$

$$=0.473 \text{ seconds}$$

Range

The range for a data set is depends on two values (the smallest and the largest values) among all values in such data set. The range is defined as the difference between the largest value and the lowest value.

Mean deviation

Another measure of variation is called mean deviation; it is the mean of the distances between each value and the mean.

Coefficient of variation

A coefficient of variation (CV) is one of well-known measures that used to compare the variability of two different data sets that have different units of measurement.

Moreover, one disadvantage of the standard deviation that its being a measure of absolute variability and not of relative variability.

The coefficient of variation, denoted by (CV), expresses standard deviation as a percentage of the mean and is computed as follows:

For population data $CV=\frac{\sigma}{\mu}\times 100\%$

For sample data $CV=\frac{S}{\bar{x}}\times 100\%$

Example:

Two plants C and D of a factory show the following results about the number of workers and the wages paid to them.

	C	D
No. of workers	5000	6000
Average monthly wages	$2500	$2500
Standard deviation	9	10

Using coefficient of variation, find in which plant, C or D there is greater variability in individual wages.

In which plant would you prefer to invest in?

Solution

To find which plant has greater variability, we need to find the coefficient of variation. The plant that has a higher coefficient of variation will have greater variability. Coefficient of variation for plant C:

Using coefficient of variation formula,

$$C.V. = \frac{\sigma}{\bar{x}} \times 100\%, \bar{x} \neq 0$$

$$C.V. = \frac{9}{2500} \times 100\% = 0.36\%$$

Now, Coefficient of variation for plant D:

$$C.V. = \frac{10}{2500} \times 100\% = 0.4\%$$

Plant C has CV = 0.36 and plant D has CV = 0.4

Hence plant D has greater variability in individual wages.

I would prefer to invest in plant C as it has lower coefficient (of variation) because it provides the most optimal risk-to-reward ratio with low volatility but high returns.

Application activity 4.3.2

A music school has budgeted to purchase three musical instruments. They plan to purchase a piano costing \$3,000, a guitar costing \$550, and a drum set costing \$600. The mean cost for a piano is \$4,000 with a standard deviation of \$2,500. The mean cost for a guitar is \$500 with a standard deviation of \$200. The mean cost for drums is \$700 with a standard deviation of \$100. Which cost is the lowest, when compared to other instruments of the same type? Which cost is the highest when compared to other instruments of the same type. Justify your answer.

4.3.3 Determining the position of data value using quartiles

Learning Activity 4.3.3

Consider the prices of 11 items arranged in order of rank in the table below.

Rank order	1	2	3	4	5	6	7	8	9	10	11
Prices (FRW)	4500	4600	4800	5200	5400	5600	6300	6400	7700	7700	7800

i) Identify the median price.

ii) How many items were bought at a price below the median price?

iii) How many items were bought at a price above the median price?

iv) Find the middle price of the lower half of the set of prices.

v) Find the middle price of the upper half of the set of prices.

vi) Together with the median price, what do the middle prices in (iv) and (v) do to the given data? Discuss.

CONTENT SUMMARY

Any data set can be divided into four equal parts by using a summary measure called quartiles.

There are three quartiles that used to divide the data set which is denoted by Q_i for $i = 1, 2, 3$. The following definition is illustrated the meaning of the quartiles. Quartiles are three summary measures that divide a ranked data set into four equal parts. The following are three quartiles:

- First quartile (Q_1) is the middle term among the observations that are less than the median.
- Second quartile (Q_2) is the same as the median.
- Third quartile (Q_3) is the value of the middle term among the observations that are greater than the median.

How to find quartiles

In general, the lower quartile, Q_1 takes the $\left[\frac{1}{4}(n+1)\right]^{th}$ position from the lower end on the rank order. The upper quartile, Q_3 takes the $\left[\frac{3}{4}(n+1)\right]^{th}$ position on the rank order. For large population, it is enough to use $\left[\frac{1}{4}(\text{n})\right]^{th}$ and $\left[\frac{3}{4}(n)\right]^{th}$ positions for the lower and upper quartiles respectively.

Example

For the given data set 61, 24, 39, 51, 37, 59, 45. Find the values of the three quartiles. First, we rank the given data in increasing order. Then we calculate the three quartiles as follows 24 37 39 45 51 59 61.

- Median value is 45.
- Values less than the median are 24 37 and 39.
- Values greater than the median are 51 59 and 61.

Therefore, the values of the three quartiles are $Q_1 = 37$, $Q_2 = 45$ $Q_3 = 59$.

Application activity 4.3.3

The heights in cm of 13 boys are: 163, 162, 170, 161, 165, 163, 162, 163, 164, 160, 158, 153, 165. Determine the three quartiles.

4.5 Measure of symmetry

4.5.1 Skewness

Learning Activity 4.3.4

Consider the two data sets that were recorded for the temperature: Dataset1: 78, 78, 79,77,76,72,74,75,74,75,76,77, 76; Dataset2: 66, 65, 58, 59, 61, 59, 61, 58, 60, 64, 59, 64, 60, 59, 58, 59, 61, 58, 60, 61, 58, 60, 63, 58, 60, 63, 58, 60, 63, 59.

a) Represent the two datasets using bar graphs.

b) Find the mean, median, and mode of each dataset.

c) Compare the mean, median, and mode of each dataset. Do all measures of a central tendency (mean, median, and mode) lie in the middle?

CONTENT SUMMARY

Sometimes data are distributed equally on the right and left of the mean value. Such data are said to be normally distributed or bell curved. They are also called symmetric. This means that the right and the left of the distribution are perfect mirror images of one another.

Not all data is symmetrically distributed. Sets of data that are not symmetric are said to be asymmetric. The measure of how da***ta are asymmetric o***r symmetric can be is called skewness. The mean, median and mode are all measures of the center of a set of data**. The** skewness of the data can be determined by how these quantities are related to one another. There are three types of skewness:

- Right skewness
- Zero skewness
- Left skewness.

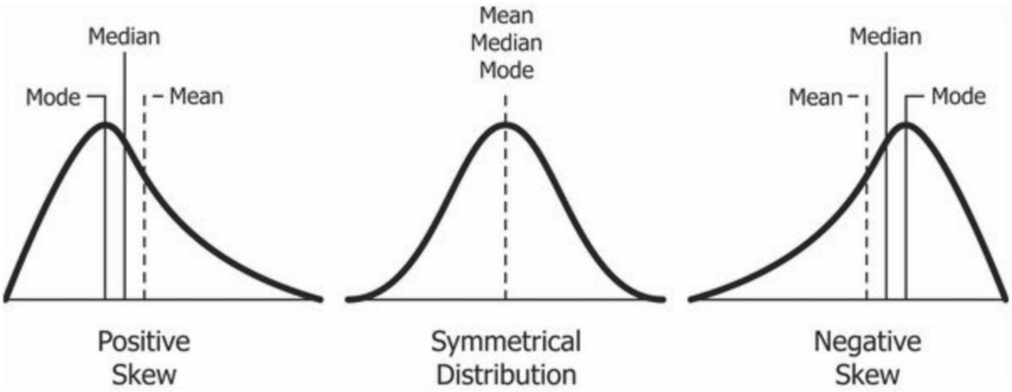


Figure 1: Three types of skewness (Positive skew, Zero skew or symmetrical distribution, and Negative skew)

Data skewed to the right (Positive skewness).

Data that are skewed to the right have a long tail that extends to the right. An alternate way of talking about a data set skewed to the right is to say that it is positively skewed. Generally, most of the time for data skewed to the right, the mean will be greater than the median and both are greater than the mode.

In summary, for a data set skewed to the right: Mode > Median >Mode.

Data skewed to the left (Negative skewness).

Data that are skewed to the left have a long tail that extends to the left. An alternate way of talking about a data set skewed to the left is to say that it is negatively skewed. Generally, most of the time for data skewed to the left, the mean will be less than the median and both less than the mode.

In summary, for a data set skewed to the left: Mode > Median >Mode.

Zero skewness

The symmetrical data has zero skewness as all measures of a central tendency lies in the middle.

In summary, for a data set skewed to the left: $Mode = Median = Mean$.

Measure of skewness

It's one thing to look at two sets of data and determine that one is symmetric while the other is asymmetric. It's another to look at two sets of asymmetric data and say that one is more skewed than the other. It can be very subjective to determine which is more skewed by simply looking at the graph of the distribution. This is why there are ways to numerically calculate the measure of skewness.

One measure of skewness, called Pearson's first coefficient of skewness, is to subtract the mean from the mode, and then divide this difference by the standard deviation of the data.

$$Skewness = \frac{Mean - Mode}{\sigma}$$

The reason for dividing the difference is so that we have a dimensionless quantity. This explains why data skewed to the right has positive skewness. If the data set is skewed to the right, the mean is greater than the mode, and so subtracting the mode from the mean gives a positive number. A similar argument explains why data skewed to the left has negative skewness.

Pearson's second coefficient of skewness is also used to measure the asymmetry of a data set. For this quantity, we subtract the mode from the median, multiply this number by three and then divide by the standard deviation.

$$Skewness = \frac{3(Median - Mode)}{\sigma}$$

Examples in real life

Incomes are skewed to the right because even just a few individuals who earn millions of dollars can greatly affect the mean, and there are no negative incomes.

Data involving the lifetime of a product, such as a brand of light bulb, are skewed to the right. Here the smallest that a lifetime can be is zero, and long lasting light bulbs will impart a positive skewness to the data.

Application activity 4.3.3

Discuss the skewness of the data represented in the histograms below.

4.5.2 Chebyshev's theorem and Empirical rule

Learning Activity 4.5.2

The prices of ten items in the supermarket are as follows:

Items	1	2	3	4	5	6	7	8	9	10
Price (in thousands (FRW))	75	74	75	72	73	72	73	74	72	74

a) Find the mean price and the standard deviation.

b) How many items with prices falling within one standard deviation from the mean?

c) How many items with prices falling within two standard deviations from the mean?

d) How many items with prices falling within three standard deviations from the mean?

CONTENT SUMMARY

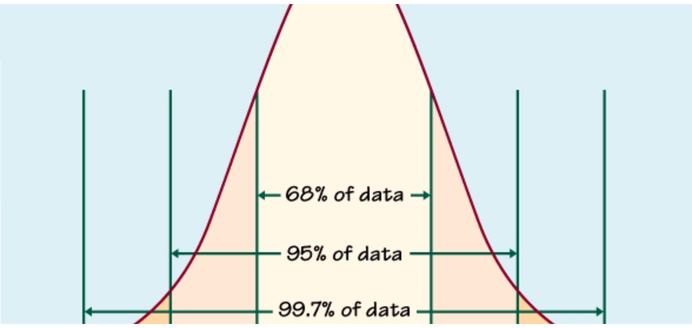
There are two ways in which we can use the standard deviation to make a statement regarding the proportion of measurements that fall within various intervals of values centered at the mean value. The information depends on the shape of histogram.

- If the histogram is bell shaped (symmetric data), the Empirical Rule is used.
- Otherwise (If data are asymmetric), Chebyshev's theorem is used.

The Empirical Rule

The Empirical Rule makes more precise statements, but it can be applied only to symmetric data (normally distributed data). For such a sample of measurements, the Empirical Rule states that:

- Approximately 68% of the observations fall in the interval $[\mu-\sigma, \mu+\sigma]$,
- Approximately 95% of the observations fall in the interval $[\mu-2\sigma, \mu+2\sigma]$
- Approximately 99.7% of the observation fall in the interval $[\mu-3\sigma, \mu+3\sigma]$



Chebyshev's theorem

Chebyshev's theorem which applies to any set of measurements (all shapes of histograms). It states that the proportion of observations that lie within k standard deviations of the mean is at least $1-\frac{1}{k^2}$, where $k>1$.
When k=2, the Chebyshev's theorem states that at least three-quarters (75%) of all observations lie within two standard deviations of the mean.

With k=3, Chebyshev's theorem states that at least eight-ninths (88.9%) of all observations lie within three standard deviations of the mean.

Application activity 4.5.2

Sweets are packed into bags with a normal mass of 75g. Ten bags are picked at random from the production line and weighed.

Their masses in grams are 76, 74.2, 75.1, 73.7, 72, 74.3, 75.4, 74, 73.1, and 72.8.

a) Find the mean mass and the standard deviation. It was later discovered that the scale was reading 3.2g below the correct weight.

b) What was the correct mean mass of the ten bags and the correct standard deviation?

c) Compare your answer in a) and b) and comment.

4.6 Examples of applications of univariate statistics in mathematical problems that involve finance, accounting, and economics.

Learning Activity 4.6

Referring to the concepts you have learnt in this unit, list down concepts and give examples of how those concepts are applied in solving problems related to finance, accounting, and economics.

CONTENT SUMMARY

With the use of descriptive statistics, we can summarize data related to revenue, expenses, and profit for companies. For example, a financial analyst who works for a retail company may calculate the following descriptive **st**atistics during **one business q**uarter:

- Mean and median number of daily sales
- Standard deviation of daily sales
- Total revenue and total expenses
- Percentage change in new customers
- Percentage of products returned by customers.
- The mean household income.
- The standard deviation of household incomes.
- The sum of gross domestic product.

- The percentage change in total new jobs.

Using these metrics, the analyst can gain a strong understanding of **the current financi**al state of the company and also compare these metrics to previous quarters to understand how the metrics are trending over time. Analyst can then use these metrics to inform the organization on areas that could use improvement to help the company increases revenue or reduce expenses.

Application activity 4.6

1. As the controller of the XXX Corporation, you are directed by the board's chairman to investigate the problem of overspending by employees with expense accounts. You ask the accounting department to provide records of the number of FRW spent by each of 25 top employees during the past month. The following record is provided: 292000, 494000, 600000, 807000, 535000, 435000, 870000, 725000, 299000, 602000, 322000, 397000, 390000, 420000, 469000, 712000, 520000, 575000, 670000, 723000, 560000, 298000, 472000, 905000, 305000. The questions the board of directors wanted to be answered are:

   a) How many of our 25 top executives spent more than 600000FRW last month?

   b) On average, how much do employees spend?

2. Consider the data on household size, annual income (in thousands (FRW)), and the number of cows for each household.

Annual income	370	490	580	680	610	640	790	890	980	950
Household size	2	4	4	1	3	5	6	4	7	2
Number of cows	0	0	1	3	2	2	1	1	1	0

Estimate the average annual income.

End of unit assessment 4

1. You are assigned by your general manager to examine each of last month's sales transactions. Find their average, find the difference between the highest and lowest sates figures, and construct a chart showing the differences between charge account and cash customers. Is this a problem in descriptive or inferential statistics?

2. When a cosmetic manufacturer tests the market to determine how many women will buy eyeliner that has been tested for safety without subjecting animals to injury, is it involved in a descriptive statistics problem or an inferential statistics problem? Explain your answer.

3. Suppose a real estate broker, is interested in the average price of a home in a development comprising 100 homes.
   a) If she uses 12 homes to predict the average price of all 100 homes, is she using inferential or descriptive statistics?
   b) If she uses all 100 homes, is she using inferential or descriptive statistics?

4. The number of sales a salesman had in the previous 7 days are: 5, 1, 2, 1, 6, 5, and 1. Calculate the variance and standard deviation.

5. Five people waiting in line at a bank were randomly chosen and asked how much cash they had in their pocket. The amounts in dollars are: 16, 17, 18, 19, and 15. Find the variance and standard deviation.

6. Refer to the table below, in which annual macroeconomic data including GDP, CPI, prime rate, private consumption, private investment, net exports, and government expenditures from 1995 to 2000 are given. Answer the following questions.
   a) How many observations are in the data set?
   b) How many variables are in the data set?
   c) Which of the variables are qualitative and which are quantitative variables?

Year	GPD (millions of FRW)	CPI	Prime rate	Private consumption (Millions of FRW)	Private investment (Millions of FRW)	Net exports (Millions of FRW)	Government expenditures (Millions of FRW)
1995	230.9	29.585	4.83	178.4	296.5	111.5	111.5
1996	296.9	29.902	4.50	182.2	294.6	119.5	119.5
1997	302.4	30.253	4.50	191.2	332.0	130.1	130.1
1998	326.7	30.633	4.50	198.9	354.3	136.4	136.4
1999	332.3	31.038	4.50	210.4	383.5	143.2	143.2
2000	3610.1	31.528	4.54	2241.8	437.3	151.4	151.4

Unit 5 Bivariate statistics and Applications

Key unit competence: Apply bivariate statistical concepts to collect, organise, analyse, present, and interpret data to draw appropriate decisions.

Introductory activity

In Kabeza village, after her 9 observations about farming, UMULISA saw that in every house observed, where there is a cow (X) if there is also domestic duck (Y), then she got the following results:

$(1,4),(2,8),(3,4),(4,12),(5,10)$
$(6,14),(7,16),(8,6),(9,18)$

a) Represent this information graphically in $(x,y)-coordinates$.

b) Find the equation of line joining any two point of the graph and guess the name of this line.

c) According to your observation from (a), explain in your own words if there is any relationship between Cows (X) and domestic duck (Y).

5.1 Introduction to bivariate statistics.

5.1.1. Key concepts of bivariate statistics

Learning Activity 5.1.1

Here is a set of data relating the temperature on days in July and the number of ice creams sold in a corner shop.

Temperature (° C)	14	16	15	16	23	12	21	22
Ice cream sales	16	18	14	19	43	12	24	26

If a businessman wants to make future investments based on how ice cream sales relate to the temperature,

i) Which statistical measure and variables will he use?

ii) Which variable depends on the other?

iii) In statistics, how do we call a variable which depends on the other?

iv) Plot the corresponding points (x, y) on a Cartesian plane and describe the resulting graph. Can you draw a line roughly representing the points in your graph?

v) Can you obtain a rule relating x and y? Explain your answer.

CONTENT SUMMARY

Bivariate data is data that has been collected in two variables, and each data point in one variable has a corresponding data **point in the other value. Bivariate data is observation on two variables, whilst univariate data is an observation on only one variable. We normally collect bivariate data to try and investigate the relationship between the two variables and then use this relationship to inform future decisions.**

Bivariate statistics deals with the collection, organization, analysis, interpretation, and drawing of conclusions from bivariate data. Data sets that contain two variables, such as wage and gender, and consumer price index and inflation rate data are said to be bivariate. In the case of bivariate or multivariate data sets we are often interested in whether elements that have high values of one of the variables also have high values of other variables.

In bivariate statistics, we have independent variable and dependent variable. Dependent variable refers to variable that depends on the other variable (s). Independent variable refers to variable that affects the other variable (s).

Scatter diagram is the graph that represents the bivariate data in x and y cartesian plane. The points do not lie on a line or a curve, hence the name. A scatter graph of bivariate data is a two-dimensional graph with one variable on one axis, and the other variable on the other axis. We then plot the corresponding points on the graph. We can then draw a regression line (also known as a line of best fit), and look at the correlation of the data (which direction the data goes, and how close to the line of best fit the data points are).For example, the scatter diagram showing the relationship between secondary school graduate rate and the % of residents who live below the poverty line.

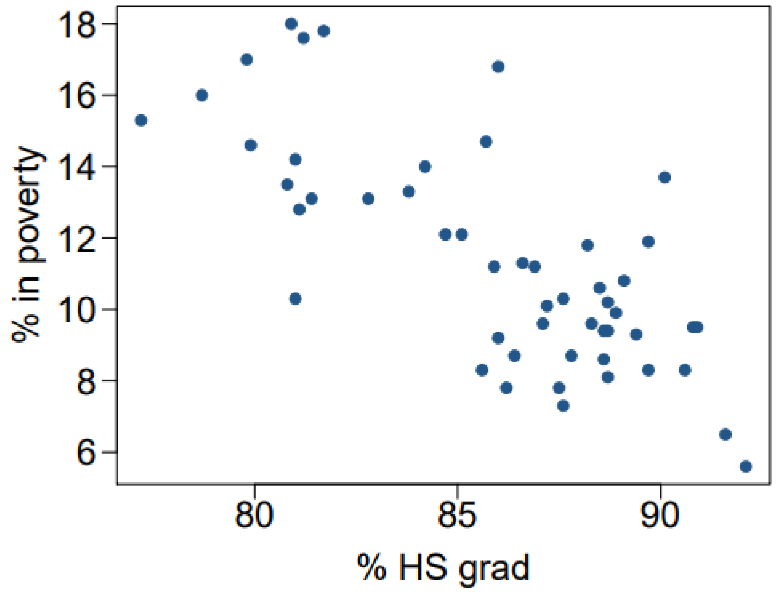


Figure 5.1: Scatter diagram representing the relationship between secondary school graduate rate and the poverty.

Examples of bivariate statistics

- Collecting the monthly savings and number of family members' data of every family that constitutes your population if you are interested in finding the relationship between savings and number of family members. In this case, you will take a small sample of families from across the country to represent the larger population of Rwanda. You will use this sample to collect data on family monthly savings and number of family members.
- We can collect data of outside temperature versus ice cream sales, these would both be examples of bivariate data. If there is a relationship showing an increase of outside temperature increased ice cream sales, then shops could use this information to buy more ice cream for hotter spells during the summer.

Application activity 5.1.1

1. Using an example, differentiate univariate statistics from bivariate statistics.
2. Using an example, differentiate dependent variable from independent variable.

5.2 Measures of linear relationship between two variables: covariance, Correlation, regression line and analysis, and spearman's coefficient of correlation.

5.2.1 Covariance and correlation.

Learning Activity 5.2.1

As students of economics, you might be interested in whether people with more years of schooling earn higher incomes. Suppose you obtain the data from one district for the population of all that district households. The data contain two variables, household income (measured in FRW) and a number of years of education of the head of each household.

i) Which statistical measure will you use to know whether people with more years of schooling earn higher incomes.

ii) If you want to know how household income and year of schooling covariate, which statistical measure will you consider?

CONTENT SUMMARY

Covariance is a statistical measure that describes the relationship between a pair of random variables where change in one variable causes change in another variable. It takes any value between -infinity to + infinity, where the negative value represents the negative relationship whereas a positive value represents the positive relationship. It is used for the linear relationship between variables. It gives the direction of relationship between variables.

How to calculate the covariance

From learning activity 5.2.1, let x_i be the value of annual household income for household i and y_i be the number of years of schooling of the head of the i^{th} household. Now consider a random sample of n households which yields the paired observations (x_i, y_i) for $i = 1, 2, 3, \ldots, n$.

The covariance of annual household income and schooling years from the learning activity 5.2.1 is given by

$$\operatorname{cov}(x,y)=\frac{1}{n}\sum_{i=1}^{n}\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right) \text{ or } \operatorname{cov}(x,y)=\frac{1}{n}\sum_{i=1}^{n}x_iy_i-\bar{x}\,\bar{y}.$$

The covariance of variables x and y is a measure of how these two variables change together. If the greater values of one variable mainly correspond with the greater values of the other variable, and the same holds for the smaller values, i.e., the variables tend to show similar behavior, the covariance is positive. In the opposite case, when the greater values of one variable mainly correspond to the smaller values of the other, i.e., the variables tend to show opposite behavior, the covariance is negative. If covariance is zero, the variables are said to be uncorrelated, meaning that there is no linear relationship between them.

Correlation

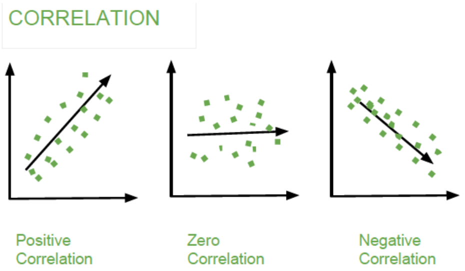
The Pearson's coefficient of correlation (or product moment coefficient of correlation or simply coefficient of correlation), denoted by r, is a measure of the strength of linear relationship between two variables. The coefficient of correlation between two variables x and y is given by $r=\dfrac{\operatorname{cov}(x,y)}{\sigma_x\sigma_y}$, where $\operatorname{cov}(x,y)$ is covariance of x and y, σ_x is the standard deviation for x, σ_y is the standard deviation for y. Correlation describes how the two variables are related. The correlation coefficient ranges from -1 to +1.

There are three types of correlation:

- Negative correlation
- Zero correlation
- Positive correlation

If the linear coefficient of correlation takes values closer to -1, the correlation is strong and negative, and will become stronger the closer r approaches -1.

If the linear coefficient of correlation takes values close to 1, the correlation is strong and positive, and will become stronger the closer r approaches 1. If the linear coefficient of correlation takes values close to 0, the correlation is weak. If r = 1 or r=-1, there is perfect correlation and the line on the scatter plot is increasing or decreasing respectively. If r = 0, there is no linear correlation.



Application activity 5.1.1

The production manager had ten newly recruited workers under him. For one week, he kept a record of the number of times that each employee needed help with a task and make a scatter diagram for the data. What type of correlation is there?

Employee	A	B	C	D	E	F	G	H	I	J
Length of employment (weeks)	24	11	47	58	3	70	76	44	33	87
Request for help	14	20	10	13	25	16	6	15	19	6

5.2.2 Regression line and analysis

Learning Activity 5.2.2

In a village, Emmanuella visited nine families and their farming activities. For each visited family there were x number of cows, and y goats. Emmanuella recorded her observations as follows:

$(1,4),(2,8),(3,4),(4,12),(5,10),(6,14),(7,16),(8,6),(9,18)$.

a) Represent Emmanuella's recorded observations graphically in cartesian plane.

b) Connect any two points on the graph drawn above to form a straight line and find equation of that line. How are the positions of the non-connected points vis-à-vis that line?

c) According to your observation from a., is there any relationship between the variation of the number of cows and the number of goats? Explain.

CONTENT SUMMARY

Bivariate statistics can help in prediction of a value for on e variable if we know the value of the other. We use the regression line to predict a value of y for any given value of x and vice versa. The "best" line wou d ma e the best predictions: the observed y-values should stray as little as possible from the line. This straight line is the regression line from which we can adjust its algebraic expressions and it is written as $y = ax + b$.

Deriving regression line equation

The regression line y on x has the form $y = ax + b$. We need the distance from this line to each point of the given data to be small, so that the sum of the square of such distances be very small.

That is $D = \sum_{i=1}^{n}\left[y_i - (ax_i + b)\right]^2$ or $D = \sum_{i=1}^{n}(y_i - ax_i - b)^2$ (1) is minimum.

1. Differentiate relation (1) with respect to b. In this case, x, y and a will be considered as constants.
2. Equate relation obtained in 1 to zero, divide each side by n and give the value of b.

- Take the value of b obtained in 2 and put it in relation obtained in 1. Differentiate the obtained relation with respect to Variance for variable x is $\sigma_x^2 = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2$
- Variance for variable y is $\sigma_y^2 = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2$
- Covariance of these two variables is $\operatorname{cov}(x, y) = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)\left(y_i - \bar{y}\right)$, give the simplified expression equal to a.

3. a, equate it to zero and divide both sides by n to find the value of a.
4. Using the relations in 2:
5. Put the value of b obtained in 2 and the value of a obtained in 4 in relation $y = ax + b$ and give the expression of regression line y on x. Hence, the regression line y on x is written as $y = \frac{\operatorname{cov}(x, y)}{\sigma_x^2}x + \left(\bar{y} - \frac{\operatorname{cov}(x, y)}{\sigma_x^2}\bar{x}\right)$.

We may write it as $L_{y/x} \equiv y-\bar{y}=\frac{\operatorname{cov}(x, y)}{\sigma_x^2}\left(x-\bar{x}\right)$.

Note that the regression line x on y is $x=cy+d$ given by

$$x-\bar{x}=\frac{\operatorname{cov}(x, y)}{\sigma_y^2}\left(y-\bar{y}\right).$$

This line is written as $L_{x/y} \equiv x-\bar{x}=\frac{\operatorname{cov}(x, y)}{\sigma_y^2}\left(y-\bar{y}\right)$.

Example

Find the regression line of y on x for the following data and estimate the value of y for $x=4, x=7, x=16$ and the value of x for $y=7, y=9, y=16$.

x	3	5	6	8	9	11
y	2	3	4	6	5	8

Answer:

x	y	$x-\bar{x}$	$y-\bar{y}$	$\left(x_i-\bar{x}\right)^2$	$\left(y_i-\bar{y}\right)^2$	$\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)$
3	2	-4	-2.6	16	6.76	10.4
5	3	-2	-1.6	4	2.56	3.2
6	4	-1	-0.6	1	0.36	0.6
8	6	1	1.4	1	1.96	1.4
9	5	2	0.4	4	0.16	0.8
11	8	4	3.4	16	11.56	13.6
$\sum_{i=1}^{6} x_i=42$	$\sum_{i=1}^{6} y_i=28$			$\sum_{i=1}^{6}\left(x_i-\bar{x}\right)^2=42$	$\sum_{i=1}^{6}\left(y_i-\bar{y}\right)^2=23.36$	$\sum_{i=1}^{6}\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)=30$

$$\bar{x}=\frac{42}{6}=7, \bar{y}=\frac{28}{6}=4.7,$$

$$(x, y)=\frac{1}{6} \sum_{i=1}^{6}\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)=\frac{30}{6},$$

$$\sigma_x^2=\frac{1}{6} \sum_{i=1}^{6}\left(x_i-\bar{x}\right)^2=\frac{42}{6}=7,$$

$$r_y^2 = \frac{1}{6}\sum_{i=1}^{6}\left(y_i - \bar{y}\right)^2 = \frac{23.36}{6} = 3.89$$

The regression line of y on x is $L_{y/x} \equiv y - 4.7 = \frac{5}{7}(x-7)$.

Application activity 5.1.1

Consider the following data:

x	60	61	62	63	65
y	3.1	3.6	3.8	4	4.1

Find the regression line of y on x and deduce the approximated value of y when x=64.

5.2.3 Spearman's coefficient of correlation

Learning Activity 5.2.2

The death rate data from 1995 to 2000 for developed and underdeveloped countries are displayed in the table below.

The death rate for developed countries (y)	2	8	4	6	5	3
The death for underdeveloped countries (x)	3	11	6	8	9	5

i) Write the death rate for developed countries and the death rate for underdeveloped countries in ascending order.

ii) Rank the death rate for developed countries and the death rate for underdeveloped countries such that the lowest death rate is ranked 1 and the highest death rate is ranked 6.

iii) Complete the table below using the information above.

Year	x	y	Rank (x)	Rank (y)	Rank (x)- Rank (y)=d	d^2
1995						
1996						
1997						
1998						
1999						
2000						
n=6						$\sum_{i=1}^{n} d_i^2 = ?$

iv) Using the information completed in the table above, find $1-\frac{6\sum_{i=1}^{n} d_i^2}{n\left(n^2-1\right)}$

CONTENT SUMMARY

A Spearman coefficient of rank correlation or Spearman's rho is a measure of statistical dependence between two variables. It assesses how well the relationship between two variables can be described using a monotonic function. The Spearman's coefficient of rank correlation is denoted and defined by $\rho = 1-\frac{6\sum_{i=1}^{n} d_i^2}{n\left(n^2-1\right)}$, where, d refers to the difference of ranks between paired items in two series and n is the number of observations.

It is much easier to calculate the Spearman's coefficient of rank correlation than to calculate the Pearson's coefficient of correlation as there is far less working involved. However, in general, the Pearson's coefficient of correlation is a more accurate measure of correlation.

Application activity 5.2.3

Calculate Spearman's coefficient of rank correlation for the series.

Income (X in thousands (FRW))	12	8	16	12	7	10	12	16	12	9
Expenditure (Y in thousands (FRW))	6	5	7	7	4	6	8	13	10	10

5.2.4 Application of bivariate statistics

Learning Activity 5.2.4

Referring to what you have learnt in this unit, discuss how bivariate statistics is used in our daily life.

CONTENT SUMMARY

Bivariate statistics can help in prediction of the value for one variable if we know the value of the other. Bivariate data occur all the time in real-world situations and we typically use the following methods to analyze the bivariate data:

- Scatter plots
- Correlation Coefficients
- Simple Linear Regression

The following examples show different scenarios where bivariate data appears in real life.

Businesses often collect bivariate data about total money spent on advertising and total revenue. For example, a business may collect the following data for 12 consecutive sales quarters:

Advertising Spend	Total Revenue
$14,500	$59,000
$19,000	$64,000
$22,400	$89,000
$28,900	$86,000
$30,000	$94,000
$32,000	$104,000
$29,000	$89,000
$28,000	$82,000
$32,000	$88,000
$35,000	$103,000
$29,000	$94,000
$38,000	$140,000

This is an example of bivariate data because it contains information on exactly two variables: advertising spend and total revenue. The business may decide to fit a simple linear regression model to this dataset and find the following fitted model:

Total Revenue = 14,942.75 + 2.70*(Advertising Spend). This tells the business that for each additional dollar spent on advertising, total revenue increases by an average of $2.70.

- Economists often collect bivariate data to understand the relationship between two socioeconomic variables. For example, an economist may collect data on the total years of schooling and *total annual income among individuals in a certain city:*

Years of Schooling	Annual Income
12	$36,000
11	$32,000
16	$58,000
16	$65,000
16	$76,000
18	$89,000
17	$45,000
20	$84,000
17	$125,000
...	...

*He may then deci*de to fit the following simple linear regression model:

Annual Income = -45,353 + 7,120*(Years of Schooling). This tells the economist that for each additional year of schooling, annual income increases by $7,120 on average.

- Biologists often collect bivariate data to understand how two variables are related among plants or animals. For example, a biologist may collect data on total rainfall and total number of plants in different regions:

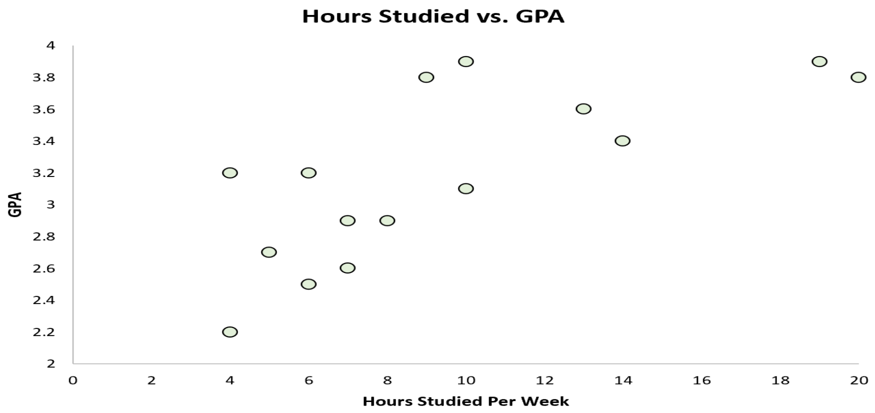
Total Rainfall (inches)	Total Number of Plants
14	450
12	413
20	490
22	566
24	576
29	640
13	340
6	130
11	190
...	...

The biologist may then decide to calculate the correlation between the two variables and find it to be 0.926. This indicates that there is a strong positive correlation between the two variables. That is, higher rainfall is closely associated with an increased number of plants in a region.

- Researchers often collect bivariate data to understand what variables affect the performance of students. For example, a researcher may collect data on the number of hours studied per week and the corresponding GPA for students in a certain class:

Hours	GPA
6	3.2
9	3.8
10	3.9
13	3.6
6	2.5
5	2.7
7	2.9
19	3.9
20	3.8
14	3.4
10	3.1
7	2.6
4	2.2
8	2.9
4	3.2

She may then create a simple scatter plot to visualize the relationship between these two variables:



Clearly there is a positive association between the two variables: As the number of hours studied per week increases, the GPA of the student tends to increase as well.

With the use of bivariate data we can quantify the relationship between variables related to promotions, advertising, sales, and other variables.

Time series forecasting allows financial analysts to predict future revenue, expenses, new customers, sales, etc. for a variety of companies.

Application activity 5.2.3

A company is to replace its fleet of cars. Eight possible models are considered and the transport manager is asked to rank them, from 1 to 8, in order of preference. A saleswoman is asked to use each type of car for a week and grade them according to their suitability for the job (A-very suitable to E-unsuitable). The price is also recorded:

Model	S	T	U	V	W	X	Y	Z
Transport manager's ranking	5	1	7	2	8	6	4	3
Saleswoman's grade	B	B+	D-	C	B+	D	C+	A-
Price (£10s)	611	811	591	792	520	573	683	716

a) Calculate the Spearman's coefficient of rank correlation between:

i) Price and transport manager's rankings,

ii) Price and saleswoman's grades.

b) Based on the result of a, state, giving a reason, whether it would be necessary to use all the three different methods of assessing the cars.

c) A new employee is asked to collect further data and to do some calculations. He produces the following results: The coefficient of correlation between

i) Price and boot capacity is 1.2,

ii) Maximum speed and fuel consumption in miles per gallons is -0.7,

iii) Price and engine capacity is -0.9. For each of his results, say giving a reason, whether you think it is reasonable.

d) Suggest two sets of circumstances where Spearman's coefficient of rank correlation would be preferred to the Pearson's coefficient of correlation as a measure of association.

End of unit assessment 5

1. Table below shows the marks awarded to six students in accounting competition:

Student	A	B	C	D	E	F
Judge1	6.8	7.3	8.1	9.8	7.1	9.2
Judge2	7.8	9.4	7.9	9.6	8.9	6.9

Calculate a coefficient of rank correlation.

2. At the end of a season, a league of eight hockey clubs produced the following table showing the position of each club in the league and the average attendance (in hundreds) at home matches.

Club	A	B	C	D	E	F	G	H
Position	1	2	3	4	5	6	7	8
Average position	27	29	9	16	24	15	12	22

Calculate Spearman's coefficient of rank correlation between the position in the league and average attendance. Comment on your results.

3. The following results were obtained from lineups in Accounting and Finance examinations:

	Accounting (x)	Finance (y)
Mean	475	39.5
Standard deviation	16.8	10.8
r	0.95	

Find both equations of regression lines. Also estimate the value of y for x=30.

4. The following results were obtained from records of age (x) and systolic blood pressure (y) of a group of 10 men:

	(x)	(y)
Mean	53	142
Variance	130	165

$$\sum_{i=1}^{n}\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)=1220$$

Find both equations of the regression lines. Also estimate the blood pressure of a man whose age is 45.

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